

Linear Inviscid Damping for Couette Flow in Stratified Fluid

Jincheng Yang

Department of Chemical Engineering
Xi'an Jiaotong University
Xi'an, 710049, China

Zhiwu Lin

School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332, USA

Abstract

We study the inviscid damping of Couette flow with an exponentially stratified density. The optimal decay rates of the velocity field and density are obtained for general perturbations with minimal regularity. For Boussinesq approximation model, the decay rates we get are consistent with the previous results in the literature. We also study the decay rates for the full equations of stratified fluids, which were not studied before. For both models, the decay rates depend on the Richardson number in a very similar way. Besides, we also study the inviscid damping of perturbations due to the exponential stratification when there is no shear.

1 Introduction

Couette flow in exponentially stratified fluid is a shear flow $U(y) = Ry$ with the density profile $\rho_0(y) = Ae^{-\beta y}$. The stability of such flow was first studied by Taylor (1931) in the half space by the method of normal modes. He present convincing but somewhat incomplete analysis to show that the spectrum of the linearized equation (now called Taylor-Goldstein equation) is quite different when the Richardson number $B^2 = \frac{\beta g}{R^2}$ (g is gravitational constant) is greater or less than $1/4$. He found that there exist infinitely many discrete neutral eigenvalues when $B^2 > \frac{1}{4}$ and no such neutral waves exist when $B^2 < \frac{1}{4}$. This claim was later proved by Dyson (1960) and Dikki (1960b). However, Taylor did not provide a clear answer to the problem of stability of Couette flow. From 1950s, there have been lots of work trying to understand the stability of stratified Couette flow, by studying the initial value problem. They include Hoiland (1952), Eliassen et al. (1953), Case (1960b), Dikki (1960a), Kuo (1963),

Hartman (1975), Chimonas (1979), Brown and Stewartson (1979), Farrel and Ioannou (1993). We refer to Section 3.2.3 of Yaglom (2012) for a detailed survey of the literature. Most of the papers used the Boussinesq approximation. One exception is Dikii (1960a), where he proved the Liapunov stability of Couette flow in the half space for the full stratified Euler equations, and for any $B^2 > 0$. We note that for the exponentially stratified fluid (i.e. $\rho_0(y) = Ae^{-\beta y}$), the Boussinesq approximation is valid only when β is small. One interesting result following from the initial value approach is the inviscid damping of velocity fields. Such inviscid damping phenomena was first known in Orr (1907), where the Couette flow in a homogeneous fluid was considered. Orr showed that the horizontal and vertical velocity decays by t^{-1} and t^{-2} respectively. Such damping is not due to the dissipation, while is by the mixing of the vorticity under the Couette flow. Recently, the inviscid damping phenomena attracted new attention. In Lin and Zeng (2011), it was shown that if we consider initial (vorticity) perturbation in the Sobolev space H^s ($s < \frac{3}{2}$) then the nonlinear damping is not true due to the existence of nonparallel steady flows of the form of Kelvin's cats eye near Couette. In Bedrossian and Masmoudi (2015), the nonlinear inviscid damping was proved for perturbations in Gevrey class (i.e. almost analytic).

In this paper, our goal is to get the precise estimates of linear decay rates for Couette flow in exponentially stratified fluid, which might be useful for the future study of nonlinear damping. We restrict ourselves to the case in the whole space. The including of the boundary (half space, finite channel) causes additional complication, as can be seen from Taylor's results mentioned at the beginning.

Our first result is about the linear decay estimates for solutions of the linearized equations under Boussinesq approximation.

Theorem 1 *Consider the steady shear flow with $U(y) = Ry$, $\rho_0(y) = Ae^{-\beta y}$, where $R \in \mathbb{R}, A > 0, \beta > 0$ are constants. Let $\psi^0(x, y) = \psi(0; x, y)$ and $\rho^0(x, y) = \frac{\rho(0; x, y)}{\rho_0(y)}$ be the initial perturbations of the stream function and relative density variation. Denote $B^2 = \frac{\beta g}{R^2}$ to be the Richardson number. Below, $f \lesssim g$ stands for $f \leq Cg$ for a constant C depending only on R, β, g . We denote $\langle f \rangle := \sqrt{1 + f^2}$ and $P_{\neq 0}$ to be the projection to nonzero Fourier modes (in x), that is,*

$$P_{\neq 0}f(t; x, y) = f(t; x, y) - \frac{1}{2\pi} \int_{\mathbb{T}} f(t; x, y) dx.$$

Let $(\mathbf{v} = \nabla^\perp \psi = (v^x, v^y), \rho)$ be the solution of the linearized equations (12)-(13) with initial data $\psi^0(x, y)$ and $\rho(0; x, y)$, then the following is true:

(i) If $0 < B^2 < \frac{1}{4}$, let $\nu = \sqrt{\frac{1}{4} - B^2}$, then

$$\begin{aligned}\|P_{\neq 0}v^x\|_{L^2} &\lesssim \langle t \rangle^{-\frac{1}{2}+\nu} \left(\|\psi^0\|_{H_x^1 H_y^2} + \|\rho^0\|_{L_x^2 H_y^1} \right), \\ \|v^y\|_{L^2} &\lesssim \langle t \rangle^{-\frac{3}{2}+\nu} \left(\|\psi^0\|_{H_x^1 H_y^3} + \|\rho^0\|_{L_x^2 H_y^2} \right), \\ \|P_{\neq 0}\rho/\rho_0\|_{L^2} &\lesssim \langle t \rangle^{-\frac{1}{2}+\nu} \left(\|\psi^0\|_{H_x^1 H_y^2} + \|\rho^0\|_{L_x^2 H_y^1} \right).\end{aligned}$$

(ii) If $B^2 > \frac{1}{4}$ then

$$\begin{aligned}\|P_{\neq 0}v^x\|_{L^2} &\lesssim \langle t \rangle^{-\frac{1}{2}} \left(\|\psi^0\|_{H_x^1 H_y^2} + \|\rho^0\|_{L_x^2 H_y^1} \right), \\ \|v^y\|_{L^2} &\lesssim \langle t \rangle^{-\frac{3}{2}} \left(\|\psi^0\|_{H_x^1 H_y^3} + \|\rho^0\|_{L_x^2 H_y^2} \right), \\ \|P_{\neq 0}\rho/\rho_0\|_{L^2} &\lesssim \langle t \rangle^{-\frac{1}{2}} \left(\|\psi^0\|_{H_x^1 H_y^2} + \|\rho^0\|_{L_x^2 H_y^1} \right).\end{aligned}$$

(iii) If $B^2 = \frac{1}{4}$, then

$$\begin{aligned}\|P_{\neq 0}v^x\|_{L^2} &\lesssim \langle t \rangle^{-\frac{1}{2}} \langle \log \langle t \rangle \rangle \left(\|\psi^0\|_{H_x^1 H_y^2} + \|\rho^0\|_{L_x^2 H_y^1} \right), \\ \|v^y\|_{L^2} &\lesssim \langle t \rangle^{-\frac{3}{2}} \langle \log \langle t \rangle \rangle \left(\|\psi^0\|_{H_x^1 H_y^3} + \|\rho^0\|_{L_x^2 H_y^2} \right), \\ \|P_{\neq 0}\rho/\rho_0\|_{L^2} &\lesssim \langle t \rangle^{-\frac{1}{2}} \langle \log \langle t \rangle \rangle \left(\|\psi^0\|_{H_x^1 H_y^2} + \|\rho^0\|_{L_x^2 H_y^1} \right).\end{aligned}$$

(iv) If $B^2 = 0$, i.e., $\beta = 0$, then $\left\| \frac{\rho}{\rho_0} \right\|_{L^2}(t) = \|\rho^0\|_{L^2}$ and

$$\begin{aligned}\|P_{\neq 0}v^x\|_{L^2} &\lesssim \|\rho^0\|_{L_x^2 H_y^1} + \langle t \rangle^{-1} \|\psi^0\|_{H_x^1 H_y^3}, \\ \|v^y\|_{L^2} &\lesssim \langle t \rangle^{-1} \|\rho^0\|_{L_x^2 H_y^2} + \langle t \rangle^{-2} \|\psi^0\|_{H_x^1 H_y^4}.\end{aligned}$$

(v) If $B^2 = \infty$, i.e. $R = 0$, then the quantity

$$\frac{g}{\beta} \left\| \frac{\rho}{\rho_0} \right\|_{L^2}^2 + \|\mathbf{v}\|_{L^2}^2$$

is conserved. The following decay estimates hold true in $L_x^2 L_y^\infty$,

$$\begin{aligned}\|P_{\neq 0}v^x\|_{L_x^2 L_y^\infty} &\lesssim |t|^{-\frac{1}{3}} \left(\|\psi^0\|_{H_x^{3/2}(H_y^{9/2} \cap W_y^{1,1})} + \|\rho^0\|_{H_x^{1/2}(H_y^{9/2} \cap W_y^{1,1})} \right), \\ \|v^y\|_{L_x^2 L_y^\infty} &\lesssim |t|^{-\frac{1}{3}} \left(\|\psi^0\|_{H_x^{5/2}(H_y^{7/2} \cap L_y^1)} + \|\rho^0\|_{H_x^{3/2}(H_y^{7/2} \cap L_y^1)} \right), \\ \|P_{\neq 0}\rho/\rho_0\|_{L_x^2 L_y^\infty} &\lesssim |t|^{-\frac{1}{3}} \left(\|\psi^0\|_{H_x^{5/2}(H_y^{9/2} \cap W_y^{1,1})} + \|\rho^0\|_{H_x^{3/2}(H_y^{7/2} \cap L_y^1)} \right).\end{aligned}$$

Theorem 1 gives a complete picture of the linear damping for the Couette flow in exponentially stratified fluid in the whole space. More specifically, we obtain optimal decay rates for initial perturbations of minimal regularity. We make some comments to relate our results to the previous papers on this problem. When $B^2 > \frac{1}{4}$, the decay rates $t^{-\frac{3}{2}}$ for v^y and $t^{-\frac{1}{2}}$ for v^x were obtained by Booker and Bretherton (1967) for a special class of solutions, which generalized the earlier results in Phillips (1966), Chap. 5 for $B^2 \gg 1$. In Hartman (1975), the decay rates as in Theorem 1 (i)-(iii) were obtained for special solutions by hypergeometric functions, which are similar to g_1, g_2 defined in (24) and (25). However, it was not shown that general solutions can be expressed by these special solutions. Chimonas (1979) considered the case $B^2 < \frac{1}{4}$ and wrongly claimed that v^y decays at the rate $t^{2\nu-1}$ and v^x grows by $t^{2\nu}$. Later, an error in Chimonas (1979) was indicated by Brown and Stewartson (1980), where they also found the correct decay rates as in Theorem 1. In Brown and Stewartson (1980), the initial value problem was solved by taking the Laplace transform to time and then the decay rates were obtained by the asymptotic analysis of the inverse Laplace transform of the solutions. It appears that the initial data should be analytic to justify certain steps concerning branch points in their analysis. Moreover, it was assumed in Brown and Stewartson (1980) that the discrete neutral eigenvalues do not exist such that there are no poles in their solutions by Laplace transform. Our results on the decay estimates show that such discrete neutral eigenvalues indeed can not exist for any $B^2 > 0$. This contrasts significantly with the case in the half space (Taylor (1931), Dyson (1960), Dikki (1960b)) or in a finite channel (Eliassen et al. (1953)), where it was shown that there exist infinitely many discrete neutral eigenvalues when $B^2 > \frac{1}{4}$. In Theorem 1, the regularity requirement for the optimal decay rates is minimal. In particular, when $B^2 < \infty$ it suffices to have the initial vorticity $\omega(0) \in H^1$ to get the optimal decay for $\|v^x\|_{L^2}$, and $\omega(0) \in H^2$ to get the optimal decay for $\|v^y\|_{L^2}$. These minimal regularity requirement on the initial data are consistent with the results in Lin and Zeng (2011) for the Couette flow with constant density. The decay estimates in the case $B^2 = \infty$ (i.e. no shear flow) appears to not have been considered in the literature. Here, the decay mechanisms are very different for the cases of $B^2 = \infty$ and $B^2 < \infty$. When $B^2 < \infty$, the decay of $\|\mathbf{v}\|_{L^2}$ is due to the mixing of vorticity caused by the shear motion. When $B^2 = \infty$, $\|\mathbf{v}\|_{L^2}$ does not decay and the decay of $\|\mathbf{v}\|_{L^\infty}$ is due to dispersive effects of the linear waves in a stably stratified fluid.

Most papers on Couette flow used the Boussinesq approximation to analyze the linearized solutions. However, this approximation is valid only when β is small. For β not small, we should use the full Euler equations. For the linearized full Euler equations without Boussinesq approximation, we obtain similar results on decay estimates in the $e^{-\frac{1}{2}\beta y}$ weighted norms.

Theorem 2 *Consider the case $U(y) = Ry$, $\rho_0(y) = Ae^{-\beta y}$, where $R \in \mathbb{R}$, $A > 0$, $\beta > 0$ are real constants. Let $\psi^0(x, y) = \psi(0; x, y)$ and $\rho^0(x, y) = \frac{\rho(0; x, y)}{\rho_0(y)}$ be the initial perturbations of the stream function and relative density variation. Let $(\mathbf{v} = \nabla^\perp \psi, \rho)$ be the solution of the linearized equations (7)-(8) with initial*

data $\psi^0(x, y)$ and $\rho(0; x, y)$, then the following is true:

(i) If $0 < B^2 < \frac{1}{4}$, let $\nu = \sqrt{\frac{1}{4} - B^2}$, then

$$\begin{aligned}\|e^{-\frac{1}{2}\beta y} P_{\neq 0} v^x\|_{L^2} &\lesssim \langle t \rangle^{-\frac{1}{2}+\nu} \left(\|e^{-\frac{1}{2}\beta y} \psi^0\|_{H_x^1 H_y^2} + \|e^{-\frac{1}{2}\beta y} \rho^0\|_{L_x^2 H_y^1} \right), \\ \|e^{-\frac{1}{2}\beta y} v^y\|_{L^2} &\lesssim \langle t \rangle^{-\frac{3}{2}+\nu} \left(\|e^{-\frac{1}{2}\beta y} \psi^0\|_{H_x^1 H_y^3} + \|e^{-\frac{1}{2}\beta y} \rho^0\|_{L_x^2 H_y^2} \right), \\ \|e^{-\frac{1}{2}\beta y} P_{\neq 0} \rho / \rho_0\|_{L^2} &\lesssim \langle t \rangle^{-\frac{1}{2}+\nu} \left(\|e^{-\frac{1}{2}\beta y} \psi^0\|_{H_x^1 H_y^2} + \|e^{-\frac{1}{2}\beta y} \rho^0\|_{L_x^2 H_y^1} \right).\end{aligned}$$

(ii) If $B^2 > \frac{1}{4}$ then

$$\begin{aligned}\|e^{-\frac{1}{2}\beta y} P_{\neq 0} v^x\|_{L^2} &\lesssim \langle t \rangle^{-\frac{1}{2}} \left(\|e^{-\frac{1}{2}\beta y} \psi^0\|_{H_x^1 H_y^2} + \|e^{-\frac{1}{2}\beta y} \rho^0\|_{L_x^2 H_y^1} \right), \\ \|e^{-\frac{1}{2}\beta y} v^y\|_{L^2} &\lesssim \langle t \rangle^{-\frac{3}{2}} \left(\|e^{-\frac{1}{2}\beta y} \psi^0\|_{H_x^1 H_y^3} + \|e^{-\frac{1}{2}\beta y} \rho^0\|_{L_x^2 H_y^2} \right), \\ \|e^{-\frac{1}{2}\beta y} P_{\neq 0} \rho / \rho_0\|_{L^2} &\lesssim \langle t \rangle^{-\frac{1}{2}} \left(\|e^{-\frac{1}{2}\beta y} \psi^0\|_{H_x^1 H_y^2} + \|e^{-\frac{1}{2}\beta y} \rho^0\|_{L_x^2 H_y^1} \right).\end{aligned}$$

(iii) If $B^2 = \frac{1}{4}$, then

$$\begin{aligned}\|e^{-\frac{1}{2}\beta y} P_{\neq 0} v^x\|_{L^2} &\lesssim \langle t \rangle^{-\frac{1}{2}} \langle \log \langle t \rangle \rangle \left(\|e^{-\frac{1}{2}\beta y} \psi^0\|_{H_x^1 H_y^2} + \|e^{-\frac{1}{2}\beta y} \rho^0\|_{L_x^2 H_y^1} \right), \\ \|e^{-\frac{1}{2}\beta y} v^y\|_{L^2} &\lesssim \langle t \rangle^{-\frac{3}{2}} \langle \log \langle t \rangle \rangle \left(\|e^{-\frac{1}{2}\beta y} \psi^0\|_{H_x^1 H_y^3} + \|e^{-\frac{1}{2}\beta y} \rho^0\|_{L_x^2 H_y^2} \right), \\ \|e^{-\frac{1}{2}\beta y} P_{\neq 0} \rho / \rho_0\|_{L^2} &\lesssim \langle t \rangle^{-\frac{1}{2}} \langle \log \langle t \rangle \rangle \left(\|e^{-\frac{1}{2}\beta y} \psi^0\|_{H_x^1 H_y^2} + \|e^{-\frac{1}{2}\beta y} \rho^0\|_{L_x^2 H_y^1} \right).\end{aligned}$$

(iv) If $B^2 = 0$, i.e., $\beta = 0$, then the results are the same as in the Boussinesq case.

(v) If $B^2 = \infty$, i.e., $R = 0$, then

$$\left\| e^{-\frac{1}{2}\beta y} \mathbf{v} \right\|_{L^2}^2 + \frac{g}{\beta} \left\| e^{-\frac{1}{2}\beta y} \frac{\rho}{\rho_0} \right\|_{L^2}^2$$

is conserved and

$$\begin{aligned}\|e^{-\frac{1}{2}\beta y} P_{\neq 0} v^x\|_{L_x^2 L_y^\infty} &\lesssim |t|^{-\frac{1}{3}} \left(\|e^{-\frac{1}{2}\beta y} \psi^0\|_{H_x^{3/2} (H_y^{9/2} \cap W_y^{1,1})} + \|e^{-\frac{1}{2}\beta y} \rho^0\|_{H_x^{1/2} (H_y^{9/2} \cap W_y^{1,1})} \right), \\ \|e^{-\frac{1}{2}\beta y} v^y\|_{L_x^2 L_y^\infty} &\lesssim |t|^{-\frac{1}{3}} \left(\|e^{-\frac{1}{2}\beta y} \psi^0\|_{H_x^{5/2} (H_y^{7/2} \cap L_y^1)} + \|e^{-\frac{1}{2}\beta y} \rho^0\|_{H_x^{3/2} (H_y^{7/2} \cap L_y^1)} \right), \\ \|e^{-\frac{1}{2}\beta y} P_{\neq 0} \rho / \rho_0\|_{L_x^2 L_y^\infty} &\lesssim |t|^{-\frac{1}{3}} \left(\|e^{-\frac{1}{2}\beta y} \psi^0\|_{H_x^{5/2} (H_y^{9/2} \cap W_y^{1,1})} + \|e^{-\frac{1}{2}\beta y} \rho^0\|_{H_x^{3/2} (H_y^{7/2} \cap L_y^1)} \right).\end{aligned}$$

Compared with Theorem 1, it is interesting to note that for the $e^{-\frac{1}{2}\beta y}$ weighted v and ρ , the decay rates and the initial regularity requirement for the full equations are exactly the same as in the Boussinesq approximation.

Lastly, we make some comments on the proof. First, we use Fourier transform for the linearized equations in the shear coordinates and then reduce them

to a second order ODE for the stream function. The general solution is expressed by two special solutions of hypergeometric functions. In this process, two new identities of hypergeometric functions (Lemmas 5 and 6) play important roles. The decay rates and initial regularity are then obtained by using the asymptotic behaviours of hypergeometric functions. In the case of $B^2 = \infty$ (no shear), the decay rates are obtained by the dispersive estimates and oscillatory integrals.

2 Preliminary

2.1 Linearized Euler Equation and Boussinesq Approximation

The equations for two dimensional inviscid incompressible flows in stratified fluid are

$$\rho (\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p = \rho \mathbf{g}, \quad (1)$$

$$(\partial_t + \mathbf{v} \cdot \nabla) \rho = 0, \quad (2)$$

$$\nabla \cdot \mathbf{v} = 0,$$

where $(x, y) \in \mathbb{T} \times \mathbb{R}$, $\mathbf{v} = (v^x, v^y)$ is the velocity, ρ is the density and $\mathbf{g} = (0, -g)$ is the gravitational acceleration directing downward with g being the gravitational constant. The simplest stationary solution is the shear flow, with $\mathbf{v}_0 = (U(y), 0)$ and $\rho_0 = \rho_0(y)$. Let $\psi = \psi(t; x, y)$ be the stream function such that $\mathbf{v} = \nabla^\perp \psi$. Here $\nabla^\perp = (-\partial_y, \partial_x)$.

We consider the linearized equations near a shear (\mathbf{v}_0, ρ_0) . Let $\mathbf{v} = \nabla^\perp \psi$ and ρ be infinitesimal perturbations of velocity and density. Here, ψ is the stream function. The linearized equations are

$$\rho_0 [(\partial_t + U(y)\partial_x) \mathbf{v} + (v^y \partial_y) \mathbf{v}_0] + \nabla p = \rho \mathbf{g}, \quad (3)$$

$$(\partial_t + U(y)\partial_x) \rho + v^y \partial_y \rho_0 = 0. \quad (4)$$

$$\nabla \cdot \mathbf{v} = 0.$$

Taking the curl of (3), we get

$$\begin{aligned} -\frac{\rho'_0(y)}{\rho_0} [U'(y)\partial_x \psi + (\partial_t + U(y)\partial_x) (-\partial_y \psi)] \\ + (\partial_t + U(y)\partial_x) \Delta \psi - U''(y)\partial_x \psi = -\partial_x \left(\frac{\rho}{\rho_0} \right) g. \end{aligned} \quad (5)$$

The equation (4) can be written as

$$(\partial_t + U(y)\partial_x) \frac{\rho}{\rho_0} = -\partial_x \psi \frac{\partial_y \rho_0}{\rho_0}. \quad (6)$$

Consider Couette flow with an exponential density profile, that is, $U(y) = Ry$, $\rho_0(y) = Ae^{-\beta y}$. Then (5)-(6) become

$$\beta [R\partial_x - (\partial_t + Ry\partial_x)\partial_y] \psi + (\partial_t + Ry\partial_x) \Delta\psi = -\partial_x \left(\frac{\rho}{\rho_0} \right) g, \quad (7)$$

$$(\partial_t + Ry\partial_x) \left(\frac{\rho}{\rho_0} \right) = \beta \partial_x \psi. \quad (8)$$

If $R \neq 0$, denote $B^2 = \frac{\beta g}{R^2}$ to be the Richardson number, $T = \frac{R\rho}{\beta\rho_0(y)}$ be the relative density perturbation, $\omega = -\Delta\psi$ be the vorticity perturbation and $t' = Rt$. Then we have

$$\begin{aligned} -\beta [\partial_x - (\partial_{t'} + y\partial_x)\partial_y] \psi + (\partial_{t'} + y\partial_x)\omega &= B^2 \partial_x T, \\ (\partial_{t'} + y\partial_x)T &= \partial_x \psi. \end{aligned}$$

For convenience we still use t for t' . Thus the resulting linearized system is

$$-\beta [\partial_x - (\partial_t + y\partial_x)\partial_y] \psi + (\partial_t + y\partial_x)\omega = B^2 \partial_x T, \quad (9)$$

$$(\partial_t + y\partial_x)T = \partial_x \psi, \quad (10)$$

$$\omega = -\Delta\psi. \quad (11)$$

The system (9)-(11) is rather complicated. Many authors, including Hoi-land (1952), Case (1960b), Kuo (1963), Hartman (1975), Chimonas (1979), Brown and Stewartson (1979), Farrel and Ioannou (1993), chose to consider the Boussinesq approximation, where the variation of density is ignored except for the gravity force term ρg . We note that to apply Boussinesq approximation, the density perturbation should be relatively small compare with the density. By this assumption, Euler momentum equation becomes

$$\rho_0 (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p = \rho \mathbf{g}.$$

The linearized equation by Boussinesq approximation (Brown and Stewartson, 1979) is

$$\begin{aligned} (\partial_t + U(y)\partial_x) \Delta\psi - U''(y)\partial_x \psi &= -\partial_x \left(\frac{\rho}{\rho_0} \right) g, \\ (\partial_t + U(y)\partial_x) \frac{\rho}{\rho_0} &= -\partial_x \psi \frac{\partial_y \rho_0}{\rho_0}. \end{aligned}$$

Comparing this with the linearized original equation (5), it can be regarded as the case when ρ'_0/ρ_0 is very small, such that the first term of (5) can be neglected.

For the case $U(y) = Ry$, $\rho_0 = Ae^{-\beta y}$, it becomes

$$(\partial_t + Ry\partial_x) \Delta\psi = -\partial_x \left(\frac{\rho}{\rho_0} \right) g, \quad (12)$$

$$(\partial_t + Ry\partial_x) \left(\frac{\rho}{\rho_0} \right) = \beta \partial_x \psi. \quad (13)$$

For $R \neq 0$,

$$\begin{aligned} \left(\frac{\partial_t}{R} + y\partial_x \right) \Delta \psi &= -\partial_x \left(\frac{\rho}{\rho_0} \right) g, \\ \left(\frac{\partial_t}{R} + y\partial_x \right) \left(\frac{\rho}{\rho_0} \right) &= \frac{\beta}{R} \partial_x \psi. \end{aligned}$$

Using the same notations as before, we have

$$(\partial_t + y\partial_x)\omega = B^2\partial_x T, \quad (14)$$

$$(\partial_t + y\partial_x)T = \partial_x \psi, \quad (15)$$

$$\omega = -\Delta \psi. \quad (16)$$

2.2 Sobolev spaces

Define the Fourier transform of $f(x, y)$ $((x, y) \in \mathbb{T} \times \mathbb{R})$, as

$$\hat{f}(k, \eta) = \frac{1}{2\pi} \int_{\mathbb{T} \times \mathbb{R}} e^{-ixk - iy\eta} f(x, y) dx dy, \quad (k, \eta) \in \mathbb{Z} \times \mathbb{R}.$$

Fourier inversion formula is

$$f(x, y) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} e^{ixk + iy\eta} \hat{f}(k, \eta) d\eta.$$

The Sobolev space $H_x^{s_x} H_y^{s_y}$ can be defined to be all L^2 functions f satisfying

$$\sum_{k \in \mathbb{Z}} (1 + k^2)^{s_x} \int_{\mathbb{R}} (1 + \eta^2)^{s_y} |\hat{f}(k, \eta)|^2 d\eta < +\infty,$$

with the norm

$$\|f\|_{H_x^{s_x} H_y^{s_y}} = \left(\sum_{k \in \mathbb{Z}} (1 + k^2)^{s_x} \int_{\mathbb{R}} (1 + \eta^2)^{s_y} |\hat{f}(k, \eta)|^2 d\eta \right)^{\frac{1}{2}},$$

Similarly, we define

$$\|f\|_{H_x^{s_x} W_y^{s_y, p}} = \left(\sum_{k \in \mathbb{Z}} (1 + k^2)^{s_x} \|\hat{f}(k, y)\|_{W_y^{s_y, p}}^2 \right)^{\frac{1}{2}},$$

where $W_y^{s_y, p}$ is the L^p Sobolev space in \mathbb{R} and

$$\hat{f}(k, y) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ixk} f(x, y) dx, \quad k \in \mathbb{Z}.$$

2.3 Hypergeometric Functions

Gaussian hypergeometric function $F(a, b; c; z)$ is defined by power series

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

for $|z| < 1$, where

$$(x)_n = \begin{cases} 1 & n = 0, \\ x(x+1) \cdots (x+n-1) & n > 0. \end{cases}$$

Its value $F(z)$ for $|z| \geq 1$ is defined by analytic continuation. Thus if $c, z \in \mathbb{R}$, and a, b are conjugate, then $F(a, b; c; z)$ is also real. The following lemma is known as Gauss' contiguous relations.

Lemma 3 *The derivative of $F(z) = F(a, b; c; z)$ can be expressed as*

$$\begin{aligned} \frac{dF}{dz} &= \frac{ab}{c} F(a+1, b+1; c+1; z) \\ &= \frac{c-1}{z} (F(a, b; c-1; z) - F(a, b; c; z)) \\ &= \frac{1}{c(1-z)} [(c-a)(c-b)F(a, b; c+1; z) + c(a+b-c)F(a, b; c; z)]. \end{aligned}$$

Hypergeometric function is related to Euler's hypergeometric differential equation.

Lemma 4 *Assume c is not an integer. Euler's hypergeometric differential equation*

$$z(1-z)f''(z) + [c - (a+b+1)z]f'(z) - abf(z) = 0$$

has two linear independent solutions

$$\begin{aligned} f_1(z) &= F(a, b; c; z), \\ f_2(z) &= z^{1-c} F(1+a-c, 1+b-c; 2-c; z). \end{aligned}$$

Proof of these two lemmas can be found in (Bateman 1953, pp. 57, 74).

Hypergeometric functions have a branch point at $z = 1$, and another at $z = \infty$. The default cut-line connecting two branch points is chosen as $z > 1, z \in \mathbb{R}$. Pfaff transform can relate the value of a hypergeometric functions near $z = 1$ to the value of another near $z = \infty$:

$$F(a, b; c; z) = (1-z)^{-b} F\left(c-a, b; c; \frac{z}{z-1}\right), \quad (17)$$

$$F(a, b; c; z) = (1-z)^{-b} F\left(c-a, b; c; \frac{z}{z-1}\right). \quad (18)$$

By combining these two transform we can obtain Euler transform

$$F(a, b; c; z) = (1 - z)^{c-a-b} F(c - a, c - b; c; z). \quad (19)$$

When $Re(c) > Re(a + b)$ we have the Guass formula

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \quad (20)$$

When $Re(c) < Re(a + b)$, $F(a, b; c; 1)$ is infinity.

The following two results play important role in the proof of our main theorems.

Lemma 5 *For any $\nu \in \mathbb{C}$, we have the following identity:*

$$\begin{aligned} & F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) \\ & + F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; -\frac{1}{2}; z\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; z\right) \\ & - F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; z\right) = \frac{1}{(1 - z)^2}. \end{aligned}$$

Lemma 6 *For $\beta_1 \in (0, 1)$, $\nu^2 \in (-\infty, \frac{1}{4})$,*

$$\begin{aligned} F_1(z) &= F\left(\frac{1}{2} - \beta_1 - \nu, \frac{1}{2} - \beta_1 + \nu; 2 - \beta_1; z\right), \\ F_2(z) &= F\left(\frac{1}{2} + \beta_1 - \nu, \frac{1}{2} + \beta_1 + \nu; \beta_1; z\right), \end{aligned}$$

we have the following identity

$$(1 - \beta_1 + 2z\beta_1) F_1(z) F_2(z) - z(1 - z) (F_1(z) F_2'(z) - F_1'(z) F_2(z)) = 1 - \beta_1.$$

Since the proof of these lemmas are pretty long, we leave them in the appendix.

3 Solutions by Hypergeometric functions

In this section, we apply Fourier transform on the linearized systems (14-16) based on Boussinesq approximation and (9-11) based on full Euler equations respectively. Then we reduce them to a second order ODE in t , and solve them explicitly by using hypergeometric functions. We will study these equations in the sheared coordinates $(z, y) = (x - ty, y)$ and define

$$\begin{aligned} f(t; z, y) &= \omega(t; z + ty, y) = \omega(t; x, y), \\ \phi(t; z, y) &= \psi(t; z + ty, y) = \psi(t; x, y), \\ \tau(t; z, y) &= T(t; z + ty, y) = T(t; x, y). \end{aligned}$$

3.1 Boussinesq approximation

First we study the case with Boussinesq approximation. In the new coordinates (z, y) , equations (14-16) become the following:

$$\begin{aligned}\partial_t f(t; z, y) &= (\partial_t + y\partial_x)\omega(t; x, y) = B^2\partial_x T(t; x, y) = B^2\partial_z \tau(t; z, y), \\ \partial_t \tau(t; z, y) &= (\partial_t + y\partial_x)T(t; x, y) = \partial_x \psi(t; x, y) = \partial_z \phi(t; z, y), \\ [\partial_{zz} + (\partial_y - t\partial_z)^2] \phi(t; z, y) &= \psi_{xx} + \psi_{yy} = -\omega(t; x, y) = -f(t; z, y).\end{aligned}$$

By Fourier Transform $(z, y) \rightarrow (k, \eta)$, we get

$$\begin{aligned}\hat{f}_t &= B^2(ik)\hat{\tau}, \quad \hat{\tau}_t = (ik)\hat{\phi}, \\ [(ik)^2 + (i\eta - ikt)^2] \hat{\phi} &= -\hat{f}.\end{aligned}\tag{21}$$

Differentiate (21) twice with respect to t to get

$$[(ik)^2 + (i\eta - ikt)^2] \hat{\phi}_t + 2(i\eta - ikt)(-ik)\hat{\phi} = -\hat{f}_t = -B^2(ik)\hat{\tau},\tag{22}$$

$$\begin{aligned}& [(ik)^2 + (i\eta - ikt)^2] \hat{\phi}_{tt} + 4(i\eta - ikt)(-ik)\hat{\phi}_t + 2(-ik)^2\hat{\phi} \\ &= -\hat{f}_{tt} = -B^2(ik)\hat{\tau}_t = -B^2(ik)^2\hat{\phi}.\end{aligned}$$

For fixed $k \neq 0$ and η , define $s = t - \frac{\eta}{k}$ and $s_0 = -\frac{\eta}{k}$. Then we obtain a second order linear ODE for $\hat{\phi}$

$$(1 + s^2)\hat{\phi}_{tt} + 4s\hat{\phi}_t + (2 + B^2)\hat{\phi} = 0.\tag{23}$$

First, we look for special solutions of the form $\hat{\phi}(t; k, \eta) = g(-s^2)$. Let $u = -s^2$. Then $\hat{\phi}_t = -2sg'$, and $\hat{\phi}_{tt} = 4s^2g'' - 2g'$. Equation (23) becomes

$$u(1 - u)g'' + \left(\frac{1}{2} - \frac{5}{2}u\right)g' - \frac{2 + B^2}{4}g = 0.$$

This is in the form of Euler's hypergeometric differential equation

$$x(1 - x)y'' + [c - (a + b + 1)x]y' - aby = 0,$$

with $c = \frac{1}{2}$; $a, b = \frac{3}{4} \pm \frac{\nu}{2}$, where $\nu = \sqrt{\frac{1}{4} - B^2}$. By Lemma 4, it has two linearly independent solutions

$$g_1(s) = F(a, b; c; u) = F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; -s^2\right)\tag{24}$$

$$g_2(s) = iu^{1-c}F(1 + a - c, 1 + b - c; 2 - c; u) = sF\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; -s^2\right).\tag{25}$$

Therefore, the general solutions to the equation (23) can be written as

$$\hat{\phi} = C_1 g_1(s) + C_2 g_2(s) \quad (26)$$

where C_1, C_2 are some constants depending only on (k, η) . Note that although a hypergeometric function has branch point or singularity at $z = 1$, we only need its value for $z = -s^2$ which lies on the negative real axis. So there is no any ambiguity or singularity in (26).

The coefficients C_1, C_2 are determined by the initial conditions $\psi(0; x, y)$ and $T(0; x, y)$. Let $\hat{\psi}^0(k, \eta), \hat{T}^0(k, \eta)$ be Fourier transforms of $\psi(0; x, y)$ and $T(0; x, y)$. First,

$$\hat{\phi}(0; k, \eta) = \hat{\phi}^0(k, \eta) = \hat{\psi}^0(k, \eta).$$

By equation (22),

$$\hat{f}_t = k^2(1 + s^2)\hat{\phi}_t + 2k^2 s \hat{\phi},$$

Noticing that when $t = 0$, $s = -\frac{\eta}{k} = s_0$, therefore

$$\begin{aligned} \hat{\phi}_t(0; k, \eta) &= \frac{\hat{f}_t(0; k, \eta) - 2k^2 s_0 \hat{\phi}(0; k, \eta)}{k^2(1 + s_0^2)} = \frac{B^2(ik)\hat{\tau}(0; k, \eta) - 2k^2 s_0 \hat{\phi}(0; k, \eta)}{k^2(1 + s_0^2)} \\ &= \frac{1}{1 + s_0^2} \left(\frac{iB^2}{k} \hat{\tau}^0 - 2s_0 \hat{\phi}^0 \right) = \frac{1}{1 + s_0^2} \left(\frac{iB^2}{k} \hat{T}^0 - 2s_0 \hat{\psi}^0 \right). \end{aligned}$$

Now we have a linear system for (C_1, C_2)

$$\begin{aligned} C_1 g_1(s_0) + C_2 g_2(s_0) &= \hat{\psi}^0, \\ C_1 g_1'(s_0) + C_2 g_2'(s_0) &= \frac{1}{1 + s_0^2} \left(\frac{iB^2}{k} \hat{T}^0 - 2s_0 \hat{\psi}^0 \right), \end{aligned}$$

where

$$\begin{aligned} g_1(s_0) &= F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; -s_0^2\right), \\ g_2(s_0) &= s_0 F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; -s_0^2\right), \\ g_1'(s_0) &= \frac{\frac{1}{2} - 1}{-s_0^2} \left[F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; -\frac{1}{2}; -s_0^2\right) - F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; -s_0^2\right) \right] (-2s_0) \\ &= -\frac{1}{s_0} \left[F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; -\frac{1}{2}; -s_0^2\right) - F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; -s_0^2\right) \right], \\ g_2'(s_0) &= F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; -s_0^2\right) \\ &\quad + s_0 \left(\frac{\frac{3}{2} - 1}{-s_0^2} \right) \left[F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{1}{2}; -s_0^2\right) - F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; -s_0^2\right) \right] (-2s_0) \\ &= F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{1}{2}; -s_0^2\right). \end{aligned}$$

Therefore the coefficients are

$$C_1(k, \eta) = \frac{1}{\Delta} \left[g_2'(s_0) + \frac{2s_0}{1+s_0^2} g_2(s_0) \right] \hat{\psi}^0(k, \eta) + \frac{1}{\Delta} \left[-\frac{iB^2}{1+s_0^2} g_2(s_0) \right] \frac{\hat{T}^0(k, \eta)}{k}, \quad (27)$$

$$C_2(k, \eta) = \frac{1}{\Delta} \left[-g_1'(s_0) - \frac{2s_0}{1+s_0^2} g_1(s_0) \right] \hat{\psi}^0(k, \eta) + \frac{1}{\Delta} \left[\frac{iB^2}{1+s_0^2} g_1(s_0) \right] \frac{\hat{T}^0(k, \eta)}{k}, \quad (28)$$

where

$$\begin{aligned} \Delta &= g_1(s_0)g_2'(s_0) - g_1'(s_0)g_2(s_0) \\ &= F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; -s_0^2\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{1}{2}; -s_0^2\right) \\ &\quad + F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; -\frac{1}{2}; -s_0^2\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; -s_0^2\right) \\ &\quad - F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; -s_0^2\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; -s_0^2\right) \\ &= \frac{1}{(1+s_0^2)^2}, \end{aligned}$$

is strictly positive for all $s_0 \in \mathbb{R}$. Here, in the last identity above we use Lemma 5. Thus the solution of (23) is given explicitly by

$$\hat{\phi}(t; k, \eta) = C_1(k, \eta)g_1(s) + C_2(k, \eta)g_2(s).$$

As for $\hat{\tau}$, from equation (22), for $B^2 > 0$ we have

$$\begin{aligned} \hat{\tau}(t; k, \eta) &= -\frac{ik}{B^2} \left((1+s^2)\hat{\phi}_t + 2s\hat{\phi} \right), \\ &= -\frac{ik}{B^2} \left[(1+s^2) (C_1(k, \eta)g_1'(s) + C_2(k, \eta)g_2'(s)) \right. \\ &\quad \left. + 2s (C_1(k, \eta)g_1(s) + C_2(k, \eta)g_2(s)) \right]. \end{aligned} \quad (29)$$

3.2 Full Euler Equations

Now we solve the linearized systems (9)-(11) based on the full Euler equations. With f, ϕ, τ defined at the beginning of this section, equations (9)-(11) turn into

$$-\beta [\partial_z - \partial_t (\partial_y - t\partial_z)] \phi + \partial_t f = B^2 \partial_z \tau, \quad (30)$$

$$\partial_t \tau = \partial_z \phi, \quad -[\partial_{zz} + (\partial_y - t\partial_z)^2] \phi = f.$$

By Fourier Transform $(z, y) \rightarrow (k, \eta)$, (30) becomes

$$-\beta [ik - \partial_t (i\eta - ikt)] \hat{\phi} + \hat{f}_t = B^2(ik)\hat{\tau}. \quad (31)$$

Differentiate above with respect to t to get

$$-\beta [ik\partial_t - \partial_{tt} (i\eta - ikt)] \hat{\phi} + \hat{f}_{tt} = B^2(ik)\hat{\tau}_t.$$

Substituting

$$\hat{\tau}_t = (ik)\hat{\phi}, \quad \hat{f} = -[(ik)^2 + (i\eta - ikt)^2] \hat{\phi}, \quad (32)$$

we have

$$\partial_{tt} [k^2 + (\eta - kt)^2 + \beta(i\eta - ikt)] \hat{\phi} - \beta(ik)\hat{\phi}_t + B^2 k^2 \hat{\phi} = 0.$$

Define $\chi = e^{-\frac{1}{2}\beta y}\phi$, then $\hat{\phi}(k, \eta) = \hat{\chi}(k, \eta + \frac{1}{2}i\beta)$ and the above equation implies

$$\partial_{tt} \left[k^2 + \left(\eta - \frac{1}{2}i\beta - kt \right)^2 + \beta \left(i \left(\eta - \frac{1}{2}i\beta \right) - ikt \right) \right] \hat{\chi} - \beta(ik)\hat{\chi}_t + B^2 k^2 \hat{\chi} = 0,$$

and after simplification

$$\partial_{tt} \left[\frac{1}{4}\beta^2 + k^2 + (\eta - kt)^2 \right] \hat{\chi} - i\beta k \hat{\chi}_t + B^2 k^2 \hat{\chi} = 0.$$

For $k \neq 0$, again define $s = t - \frac{\eta}{k}$, $s_0 = -\frac{\eta}{k}$, then

$$\partial_{tt} \left[\left(\frac{1}{4}\beta^2 + k^2 + k^2 s^2 \right) \hat{\chi} \right] - i\beta k \hat{\chi}_t + B^2 k^2 \hat{\chi} = 0.$$

Define $m = \sqrt{\frac{1}{4}\beta^2 + k^2}$, $\kappa = \frac{k}{m}$, $\beta_1 = \frac{\beta}{2m}$, then we have

$$\partial_{tt} [(m^2 + k^2 s^2) \hat{\chi}] - i\beta k \hat{\chi}_t + B^2 k^2 \hat{\chi} = 0,$$

$$\partial_{tt} [(1 + \kappa^2 s^2) \hat{\chi}] - 2i\beta_1 \kappa \hat{\chi}_t + B^2 \kappa^2 \hat{\chi} = 0.$$

Set $u = -i\kappa s$, then

$$\begin{aligned} -\partial_{uu} (1 - u^2) \hat{\chi} - 2\beta_1 \hat{\chi}_u + B^2 \hat{\chi} &= 0, \\ (1 - u^2) \hat{\chi}_{uu} + (2\beta_1 - 4u) \hat{\chi}_u - (2 + B^2) \hat{\chi} &= 0. \end{aligned}$$

Define $v = \frac{1-u}{2}$, then

$$v(1-v) \hat{\chi}_{vv} + (-\beta_1 + 2 - 4v) \hat{\chi}_v - (2 + B^2) \hat{\chi} = 0, \quad (33)$$

which is of the form of Euler's hypergeometric differential equation. So by Lemma 4, it has two linear independent solutions,

$$\begin{aligned} g_3(s) &= F\left(\frac{3}{2} - \nu, \frac{3}{2} + \nu; 2 - \beta_1; v\right) = F\left(\frac{3}{2} - \nu, \frac{3}{2} + \nu; 2 - \beta_1; \frac{1 + i\kappa s}{2}\right), \\ g_4(s) &= \left(\frac{1 + i\kappa s}{2}\right)^{-1+\beta_1} F\left(\frac{1}{2} + \beta_1 - \nu, \frac{1}{2} + \beta_1 + \nu; \beta_1; \frac{1 + i\kappa s}{2}\right) \\ &= \left(\frac{1}{2} + \frac{\kappa s}{2}i\right)^{-1+\beta_1} F\left(\frac{1}{2} + \beta_1 - \nu, \frac{1}{2} + \beta_1 + \nu; \beta_1; \frac{1}{2} + \frac{\kappa s}{2}i\right). \end{aligned}$$

By Euler transform,

$$\begin{aligned} g_3(s) &= \left(1 - \frac{1+i\kappa s}{2}\right)^{-1-\beta_1} F\left(\frac{1}{2} - \beta_1 - \nu, \frac{1}{2} - \beta_1 + \nu; 2 - \beta_1; \frac{1+i\kappa s}{2}\right) \\ &= \left(\frac{1}{2} - \frac{\kappa s}{2}i\right)^{-1-\beta_1} F\left(\frac{1}{2} - \beta_1 - \nu, \frac{1}{2} - \beta_1 + \nu; 2 - \beta_1; \frac{1}{2} + \frac{\kappa s}{2}i\right), \end{aligned}$$

Therefore, the general solution to equation (33) is

$$\hat{\chi} = C_3 g_3(s) + C_4 g_4(s)$$

where C_3, C_4 are constants depending only on (k, η) . Note that we only need values of g_1, g_2 at $\frac{1}{2} + \frac{\kappa s}{2}i$ ($s \in \mathbb{R}$), that is, on the line $Re(z) = \frac{1}{2}$. So the branch point at $z = 1$ will not cause any ambiguity or singularity.

The initial conditions $\psi(0; x, y)$ and $T(0; x, y)$ are used to determine the coefficients C_3, C_4 . Denote $\mu = e^{-\frac{1}{2}\beta y \tau}$, $\Psi^0 = e^{-\frac{1}{2}\beta y} \psi^0$, $\Upsilon^0 = e^{-\frac{1}{2}\beta y} T^0$, then

$$\hat{\chi}(0; k, \eta) = \hat{\phi}^0\left(k, \eta - \frac{1}{2}i\beta\right) = e^{-\widehat{\frac{1}{2}\beta y}} \psi^0 = \hat{\Psi}^0.$$

By equations (31) and (32), we have

$$\hat{\phi}_t = \frac{1}{1 + s^2 - \frac{i\beta}{k}s} \left[\left(\frac{2i\beta}{k} - 2s \right) \hat{\phi} + \frac{iB^2}{k} \hat{\tau} \right].$$

Hence

$$\begin{aligned} \hat{\chi}_t(t; k, \eta) &= \hat{\phi}_t\left(t; k, \eta - \frac{1}{2}i\beta\right) \\ &= \frac{1}{1 + \left(s + \frac{i\beta}{2k}\right)^2 - \frac{i\beta}{k}\left(s + \frac{i\beta}{2k}\right)} \left[\left(\frac{2i\beta}{k} - 2s - 2\frac{i\beta}{2k} \right) \hat{\chi} + \frac{iB^2}{k} \hat{\mu} \right] \\ &= \frac{1}{1 + |\tilde{s}|^2} \left(\frac{iB^2}{k} \hat{\chi} - 2\tilde{s} \hat{\mu} \right), \end{aligned}$$

and

$$\hat{\chi}_t(0; k, \eta) = \frac{1}{1 + |\tilde{s}_0|^2} \left(\frac{iB^2}{k} \hat{\Upsilon}^0 - 2\tilde{s}_0 \hat{\Psi}^0 \right),$$

where $\tilde{s} = s - \frac{i\beta}{2k}$, $\tilde{s}_0 = s_0 - \frac{i\beta}{2k}$.

So we have a linear system for (C_3, C_4) :

$$\begin{aligned} C_3 g_3(s_0) + C_4 g_4(s_0) &= \hat{\Psi}^0, \\ C_3 g'_3(s_0) + C_4 g'_4(s_0) &= \frac{1}{1 + |\tilde{s}_0|^2} \left(\frac{iB^2}{k} \hat{\Upsilon}^0 - 2\tilde{s}_0 \hat{\Psi}^0 \right), \end{aligned}$$

where

$$\begin{aligned} g_3(s_0) &= \left(\frac{1}{2} - \frac{\kappa s_0}{2}i\right)^{-1-\beta_1} F\left(\frac{1}{2} - \beta_1 - \nu, \frac{1}{2} - \beta_1 + \nu; 2 - \beta_1; \frac{1}{2} + \frac{\kappa s_0}{2}i\right), \\ g_4(s_0) &= \left(\frac{1}{2} + \frac{\kappa s_0}{2}i\right)^{-1+\beta_1} F\left(\frac{1}{2} + \beta_1 - \nu, \frac{1}{2} + \beta_1 + \nu; \beta_1; \frac{1}{2} + \frac{\kappa s_0}{2}i\right). \end{aligned}$$

Denote

$$\begin{aligned} F_1(z_0) &= F\left(\frac{1}{2} - \beta_1 - \nu, \frac{1}{2} - \beta_1 + \nu; 2 - \beta_1; z_0\right), \\ F_2(z_0) &= F\left(\frac{1}{2} + \beta_1 - \nu, \frac{1}{2} + \beta_1 + \nu; \beta_1; z_0\right) \end{aligned}$$

where $z_0 = \frac{1}{2} + \frac{\kappa s_0}{2}i$. Then

$$\begin{aligned} g_3(s_0) &= \bar{z}_0^{-1-\beta_1} F_1(z_0), \\ g_4(s_0) &= z_0^{-1+\beta_1} F_2(z_0), \\ g'_3(s_0) &= \bar{z}_0^{-2-\beta_1} \left[(-1 - \beta_1) \left(-\frac{\kappa i}{2}\right) F_1(z_0) + \bar{z}_0 F'_1(z_0) \left(\frac{\kappa i}{2}\right) \right] \\ &= \bar{z}_0^{-2-\beta_1} \left(\frac{\kappa i}{2}\right) [(1 + \beta_1) F_1 + \bar{z}_0 F'_1], \\ g'_4(s_0) &= z_0^{-2+\beta_1} \left[(-1 + \beta_1) \left(\frac{\kappa i}{2}\right) F_2(z_0) + z_0 F'_2(z_0) \left(\frac{\kappa i}{2}\right) \right] \\ &= z_0^{-2+\beta_1} \left(\frac{\kappa i}{2}\right) [(-1 + \beta_1) F_2 + z_0 F'_2]. \end{aligned}$$

Therefore the coefficients are

$$\begin{aligned} C_3(k, \eta) &= \frac{1}{\Delta} \left[g'_4(s_0) + \frac{2\tilde{s}_0}{1 + |\tilde{s}_0|^2} g_4(s_0) \right] \hat{\Psi}^0(k, \eta) \\ &\quad + \frac{1}{\Delta} \left[-\frac{iB^2}{1 + |\tilde{s}_0|^2} g_4(s_0) \right] \frac{\hat{\Upsilon}^0(k, \eta)}{k}, \\ C_4(k, \eta) &= \frac{1}{\Delta} \left[-g'_3(s_0) - \frac{2\tilde{s}_0}{1 + |\tilde{s}_0|^2} g_3(s_0) \right] \hat{\Psi}^0(k, \eta) \\ &\quad + \frac{1}{\Delta} \left[\frac{iB^2}{1 + |\tilde{s}_0|^2} g_3(s_0) \right] \frac{\hat{\Upsilon}^0(k, \eta)}{k} \end{aligned}$$

where

$$\begin{aligned}
\Delta &= g_3(s_0)g_4'(s_0) - g_3'(s_0)g_4(s_0) \\
&= \bar{z}_0^{-2-\beta_1} z_0^{-2+\beta_1} \left(\frac{\kappa i}{2} \right) [\bar{z}_0 ((-1 + \beta_1)F_2 + z_0 F_2') F_1 - z_0 ((1 + \beta_1)F_1 + \bar{z}_0 F_1') F_2] \\
&= \bar{z}_0^{-2-\beta_1} z_0^{-2+\beta_1} \left(\frac{\kappa i}{2} \right) [(-1 + (\bar{z}_0 - z_0)\beta_1) F_1 F_2 + z_0 \bar{z}_0 (F_1 F_2' - F_1' F_2)] \\
&= \bar{z}_0^{-2-\beta_1} z_0^{-2+\beta_1} \left(\frac{\kappa i}{2} \right) [(-1 - (2z_0 - 1)\beta_1) F_1 F_2 + z_0(1 - z_0) (F_1 F_2' - F_1' F_2)] \\
&= \bar{z}_0^{-2-\beta_1} z_0^{-2+\beta_1} \left(\frac{\kappa i}{2} \right) (\beta_1 - 1),
\end{aligned}$$

is never zero, because $|\kappa|, \beta_1 \in (0, 1)$ by definition. Here, in the last identity above we use Lemma 6. Moreover,

$$|\kappa| \geq \frac{1}{\sqrt{\frac{1}{4}\beta^2 + 1}}, 1 - \beta_1 \geq 1 - \frac{\beta/2}{\sqrt{\frac{1}{4}\beta^2 + 1}}$$

are both uniformly bounded away from zero for all integers $k \neq 0$. Hence

$$|\Delta|^{-1} = |z_0|^4 \left| \frac{\kappa}{2} \right|^{-1} (1 - \beta_1)^{-1} \lesssim \langle s_0 \rangle^4.$$

By equations (31) and (32), for $B^2 > 0$ we have

$$\hat{\tau}(t; k, \eta) = -\frac{ik}{B^2} \left[-\frac{2i\beta}{k} \hat{\phi} - \frac{i\beta}{k} s \hat{\phi}_t + (1 + s^2) \hat{\phi}_t + 2s \hat{\phi} \right],$$

and

$$\begin{aligned}
\hat{\mu}(t; k, \eta) &= \hat{\tau} \left(t; k, \eta - \frac{1}{2} i\beta \right) \\
&= -\frac{ik}{B^2} \left[-\frac{2i\beta}{k} \hat{\chi} - \frac{i\beta}{k} \left(s + \frac{i\beta}{2k} \right) \hat{\chi}_t + \left(1 + \left(s + \frac{i\beta}{2k} \right)^2 \right) \hat{\chi}_t + 2 \left(s + \frac{i\beta}{2k} \right) \hat{\chi} \right] \\
&= -\frac{ik}{B^2} \left[\left(1 + s^2 + \frac{\beta^2}{4k^2} \right) \hat{\chi}_t + 2 \left(s - \frac{i\beta}{2k} \right) \hat{\chi} \right] \\
&= -\frac{ik}{B^2} [(1 + |\tilde{s}|^2) \hat{\chi}_t + 2\tilde{s} \hat{\chi}].
\end{aligned}$$

4 Decay estimates in the case of Boussinesq approximation

In this section, we use the solution formula obtained in the last section to obtain the inviscid decay estimates in Theorem 1, for solutions of the linearized equations under Boussinesq approximation.

4.1 The case $B^2 > 0$ and $B^2 \neq \frac{1}{4}$

By expanding $g_1(s)$, $g_2(s)$, $g'_1(s_0)$, $g'_2(s_0)$ at infinity similarly as in the proof of Lemma 6 (see Appendix), we obtain the following asymptotics

$$g_1(s) = \sqrt{\pi} \left[\frac{\Gamma(\nu)}{\Gamma(-\frac{1}{4} + \frac{\nu}{2})\Gamma(\frac{3}{4} + \frac{\nu}{2})} s^{-\frac{3}{2}+\nu} + \frac{\Gamma(-\nu)}{\Gamma(-\frac{1}{4} - \frac{\nu}{2})\Gamma(\frac{3}{4} - \frac{\nu}{2})} s^{-\frac{3}{2}-\nu} \right] + O\left(|s|^{-\frac{7}{2}+Re(\nu)}\right), \quad (34)$$

$$g_2(s) = \frac{\sqrt{\pi}}{2} \left[\frac{\Gamma(\nu)}{\Gamma(\frac{1}{4} + \frac{\nu}{2})\Gamma(\frac{5}{4} + \frac{\nu}{2})} s^{-\frac{3}{2}+\nu} + \frac{\Gamma(-\nu)}{\Gamma(\frac{1}{4} - \frac{\nu}{2})\Gamma(\frac{5}{4} - \frac{\nu}{2})} s^{-\frac{3}{2}-\nu} \right] + O\left(|s|^{-\frac{5}{2}+Re(\nu)}\right), \quad (35)$$

$$g'_1(s_0) = 2\sqrt{\pi} \left[\frac{(-\frac{3}{4} + \frac{\nu}{2})\Gamma(\nu)}{\Gamma(-\frac{1}{4} + \frac{\nu}{2})\Gamma(\frac{3}{4} + \frac{\nu}{2})} s_0^{-\frac{5}{2}+\nu} + \frac{(-\frac{3}{4} - \frac{\nu}{2})\Gamma(-\nu)}{\Gamma(-\frac{1}{4} - \frac{\nu}{2})\Gamma(\frac{3}{4} - \frac{\nu}{2})} s_0^{-\frac{5}{2}-\nu} \right] + O\left(|s_0|^{-\frac{7}{2}+Re(\nu)}\right), \quad (36)$$

$$g'_2(s_0) = \sqrt{\pi} \left[\frac{(-\frac{3}{4} + \frac{\nu}{2})\Gamma(\nu)}{\Gamma(\frac{1}{4} + \frac{\nu}{2})\Gamma(\frac{5}{4} + \frac{\nu}{2})} s_0^{-\frac{5}{2}+\nu} + \frac{(-\frac{3}{4} - \frac{\nu}{2})\Gamma(-\nu)}{\Gamma(\frac{1}{4} - \frac{\nu}{2})\Gamma(\frac{5}{4} - \frac{\nu}{2})} s_0^{-\frac{5}{2}-\nu} \right] + O\left(|s_0|^{-\frac{7}{2}+Re(\nu)}\right). \quad (37)$$

For $B^2 < \frac{1}{4}$ or $> \frac{1}{4}$, ν is real or pure imaginary. We treat these cases separately.

4.1.1 The case $0 < B^2 < \frac{1}{4}$

In this case ν is a real number between 0 and $\frac{1}{2}$. By using the above asymptotics of $g_1(s)$, $g_2(s)$, we obtain bounds for coefficients of C_1, C_2 (defined in (27), (28)). Since

$$\begin{aligned} \frac{1}{\Delta} \left[g'_2(s_0) + \frac{2s_0}{1+s_0^2} g_2(s_0) \right] &\lesssim \langle s_0 \rangle^4 \langle s_0 \rangle^{-\frac{5}{2}+\nu} = \langle s_0 \rangle^{\frac{3}{2}+\nu}, \\ \frac{1}{\Delta} \left[-\frac{iB^2}{1+s_0^2} g_2(s_0) \right] &\lesssim \langle s_0 \rangle^4 \langle s_0 \rangle^{-\frac{7}{2}+\nu} = \langle s_0 \rangle^{\frac{1}{2}+\nu}, \\ \frac{1}{\Delta} \left[-g'_1(s_0) - \frac{2s_0}{1+s_0^2} g_1(s_0) \right] &\lesssim \langle s_0 \rangle^4 \langle s_0 \rangle^{-\frac{5}{2}+\nu} = \langle s_0 \rangle^{\frac{3}{2}+\nu}, \\ \frac{1}{\Delta} \left[\frac{iB^2}{1+s_0^2} g_1(s_0) \right] &\lesssim \langle s_0 \rangle^4 \langle s_0 \rangle^{-\frac{7}{2}+\nu} = \langle s_0 \rangle^{\frac{1}{2}+\nu}, \end{aligned}$$

and

$$|g_1(s)|, |g_2(s)| \lesssim \langle s \rangle^{-\frac{3}{2}+\nu},$$

so we have

$$\begin{aligned} |C_1(k, \eta)| &\lesssim \langle s_0 \rangle^{\frac{3}{2}+\nu} \left(\left| \hat{\psi}^0(k, \eta) \right| + \frac{|\hat{T}^0(k, \eta)|}{\langle s_0 \rangle |k|} \right), \\ |C_2(k, \eta)| &\lesssim \langle s_0 \rangle^{\frac{3}{2}+\nu} \left(\left| \hat{\psi}^0(k, \eta) \right| + \frac{|\hat{T}^0(k, \eta)|}{\langle s_0 \rangle |k|} \right). \end{aligned}$$

Therefore

$$\begin{aligned} \left| \hat{\phi}(t; k, \eta) \right| &= |C_1(k, \eta)g_1(s) + C_2(k, \eta)g_2(s)| \\ &\lesssim \langle s \rangle^{-\frac{3}{2}+\nu} \langle s_0 \rangle^{\frac{3}{2}+\nu} \left(\left| \hat{\psi}^0(k, \eta) \right| + \frac{|\hat{T}^0(k, \eta)|}{\langle s_0 \rangle |k|} \right). \end{aligned} \quad (38)$$

To get the decay estimates in the physical space (x, y) from above, we note that the term $\langle s \rangle^{-\frac{3}{2}+\nu}$ does not decay when $t \approx \frac{\eta}{k}$ (i.e. $s \approx 0$) and as a compensation the additional regularity of initial data is needed to ensure the decay. This is made precise in the following lemma.

Lemma 7 *Assume that there exists $a > 0$ and $b, c \in \mathbb{R}$ such that*

$$|\hat{g}(t; k, \eta)| \lesssim \langle s \rangle^{-a} \langle s_0 \rangle^b |k|^c \left| \hat{h}(k, \eta) \right|, \quad 0 \neq k \in \mathbb{Z}, \eta \in \mathbb{R}, \quad (39)$$

then

$$\|P_{\neq 0}g(t)\|_{L^2(\mathbb{T} \times \mathbb{R})} \lesssim \langle t \rangle^{-a} \|h\|_{H_x^c H_y^{b+a}}.$$

Proof. We have

$$\begin{aligned} \int_{\mathbb{R}} |\hat{g}(t; k, \eta)|^2 d\eta &= \int_{|s|=|t-\frac{\eta}{k}| \geq \frac{1}{2}|t|} |\hat{g}(t; k, \eta)|^2 d\eta + \int_{|t-\frac{\eta}{k}| \leq \frac{1}{2}|t|} |\hat{g}(t; k, \eta)|^2 d\eta \\ &= I_1 + I_2. \end{aligned}$$

By (39), directly we have

$$I_1 \lesssim \langle t \rangle^{-2a} \int_{|t-\frac{\eta}{k}| \geq \frac{1}{2}|t|} \langle s_0 \rangle^{2b} |k|^{2c} \left| \hat{h}(k, \eta) \right|^2 d\eta.$$

Since $|t - \frac{\eta}{k}| \leq \frac{1}{2}|t|$ implies $|s_0| = \left| \frac{\eta}{k} \right| \geq \frac{1}{2}|t|$, so

$$I_2 \lesssim \langle t \rangle^{-2a} \int_{|t-\frac{\eta}{k}| \leq \frac{1}{2}|t|} \langle s_0 \rangle^{2b+2a} |k|^{2c} \left| \hat{h}(k, \eta) \right|^2 d\eta.$$

So

$$\int_{\mathbb{R}} |\hat{g}(t; k, \eta)|^2 d\eta \lesssim \langle t \rangle^{-2a} \int_{\mathbb{R}} \langle s_0 \rangle^{2b+2a} |k|^{2c} \left| \hat{h}(k, \eta) \right|^2 d\eta,$$

and

$$\begin{aligned} \|P_{\neq 0}g(t)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 &= \sum_{k \neq 0} \int_{\mathbb{R}} |\hat{g}(t; k, \eta)|^2 d\eta \\ &\lesssim \langle t \rangle^{-2a} \sum_{k \neq 0} |k|^{2c} \int_{\mathbb{R}} \langle \eta \rangle^{2b+2a} \left| \hat{h}(k, \eta) \right|^2 d\eta \\ &\lesssim \langle t \rangle^{-2a} \|h\|_{H_x^c H_y^{b+a}}^2. \end{aligned}$$

■

Since the velocity perturbation

$$\begin{aligned} v^x(t; x, y) &= -\partial_y \psi(t; x, y) = (-\partial_y + t\partial_z)\phi(t; z, y), \\ v^y(t; x, y) &= \partial_x \psi(t; x, y) = \partial_z \phi(t; z, y), \end{aligned}$$

so by (38), we have

$$\begin{aligned} |\hat{v}^x(t; k, \eta)| &= |iks\hat{\phi}(t; k, \eta)| \leq \langle s \rangle^{-\frac{1}{2}+\nu} \langle s_0 \rangle^{\frac{3}{2}+\nu} \left(|k| |\hat{\psi}^0(k, \eta)| + \frac{|\hat{T}^0(k, \eta)|}{\langle s_0 \rangle} \right), \\ |\hat{v}^y(t; k, \eta)| &= |ik\hat{\phi}(t; k, \eta)| \leq \langle s \rangle^{-\frac{3}{2}+\nu} \langle s_0 \rangle^{\frac{3}{2}+\nu} \left(|k| |\hat{\psi}^0(k, \eta)| + \frac{|\hat{T}^0(k, \eta)|}{\langle s_0 \rangle} \right). \end{aligned}$$

From equation (29) we know

$$\begin{aligned} |\hat{\tau}(t; k, \eta)| &\leq \left| \frac{k}{B^2} \right| \left[(1+s^2) |C_1(k, \eta)g'_1(s) + C_2(k, \eta)g'_2(s)| \right. \\ &\quad \left. + 2|s| |C_1(k, \eta)g_1(s) + C_2(k, \eta)g_2(s)| \right] \\ &\lesssim \langle s \rangle^{-\frac{1}{2}+\nu} \langle s_0 \rangle^{\frac{3}{2}+\nu} \left(|k| |\hat{\psi}^0(k, \eta)| + \frac{|\hat{T}^0(k, \eta)|}{\langle s_0 \rangle} \right). \end{aligned}$$

So by Lemma 7,

$$\begin{aligned} \|P_{\neq 0} v^x\|_{L^2} &\lesssim \langle t \rangle^{-\frac{1}{2}+\nu} \left(\|\psi^0\|_{H_x^1 H_y^2} + \|T^0\|_{L_x^2 H_y^1} \right), \\ \|v^y\|_{L^2} &\lesssim \langle t \rangle^{-\frac{3}{2}+\nu} \left(\|\psi^0\|_{H_x^1 H_y^3} + \|T^0\|_{L_x^2 H_y^2} \right), \end{aligned}$$

and

$$\|P_{\neq 0} T(t; \cdot, \cdot)\|_{L^2} = \|P_{\neq 0} \tau(t; \cdot, \cdot)\|_{L^2} \lesssim \langle t \rangle^{-\frac{1}{2}+\nu} \left(\|\psi^0\|_{H_x^1 H_y^2} + \|T^0\|_{L_x^2 H_y^1} \right).$$

4.1.2 The case $B^2 > \frac{1}{4}$

In this case, $\nu = \sqrt{\frac{1}{4} - B^2}$ is pure imaginary. Then from (34-37), we have

$$\begin{aligned} |g_1(s)| &\lesssim \langle s \rangle^{-\frac{3}{2}}, \quad |g_2(s)| \lesssim \langle s \rangle^{-\frac{3}{2}}, \\ |g'_1(s_0)| &\lesssim \langle s_0 \rangle^{-\frac{5}{2}}, \quad |g'_2(s_0)| \lesssim \langle s_0 \rangle^{-\frac{5}{2}}. \end{aligned}$$

So by similar calculations,

$$\begin{aligned} \|P_{\neq 0} v^x\|_{L^2} &\lesssim \langle t \rangle^{-\frac{1}{2}} \left(\|\psi^0\|_{H_x^1 H_y^2} + \|T^0\|_{L_x^2 H_y^1} \right), \\ \|v^y\|_{L^2} &\lesssim \langle t \rangle^{-\frac{3}{2}} \left(\|\psi^0\|_{H_x^1 H_y^3} + \|T^0\|_{L_x^2 H_y^2} \right), \\ \|P_{\neq 0} T\|_{L^2} &\lesssim \langle t \rangle^{-\frac{1}{2}} \left(\|\psi^0\|_{H_x^1 H_y^2} + \|T^0\|_{L_x^2 H_y^1} \right). \end{aligned}$$

Since T is just ρ/ρ_0 times a positive constant, this completes the proof of Theorem 1(i)-(ii).

4.2 The case $B^2 = \frac{1}{4}$

When $B^2 = \frac{1}{4}$, $\nu = 0$, the asymptotic approximation (34) and (35) no longer holds true, but the following expansions at infinity emerge instead,

$$\begin{aligned} g_1(s) &= F\left(\frac{3}{4}, \frac{3}{4}; \frac{1}{2}; -s^2\right) = \frac{2\sqrt{\pi}}{\Gamma(-\frac{1}{4})\Gamma(\frac{3}{4})} s^{-\frac{3}{2}} \log(s) - \frac{2\sqrt{\pi}(\gamma + F(\frac{3}{4}) + 2)}{\Gamma(-\frac{1}{4})\Gamma(\frac{3}{4})} s^{-\frac{3}{2}} + O(|s|^{-\frac{7}{2}}), \\ g_2(s) &= sF\left(\frac{5}{4}, \frac{5}{4}; \frac{3}{2}; -s^2\right) = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})} s^{-\frac{3}{2}} \log(s) - \frac{\sqrt{\pi}(\gamma + F(\frac{1}{4}) + 2)}{\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})} s^{-\frac{3}{2}} + O(|s|^{-\frac{7}{2}}) \end{aligned}$$

where γ is the Euler constant, $F(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function. It can be seen that with the logarithm function, both solutions decay a little bit slower than before.

Similarly, their derivatives also have different asymptotic approximation

$$\begin{aligned} g_1'(s_0) &= -\frac{9}{4}s_0 F\left(\frac{7}{4}, \frac{7}{4}; \frac{3}{2}; -s_0^2\right) \\ &= -\frac{3\sqrt{\pi}}{\Gamma(-\frac{1}{4})\Gamma(\frac{3}{4})} s_0^{-\frac{5}{2}} \log(s_0) + \frac{3\sqrt{\pi}(\gamma + F(\frac{3}{4}) + \frac{8}{3})}{\Gamma(-\frac{1}{4})\Gamma(\frac{3}{4})} s_0^{-\frac{5}{2}} + O(|s_0|^{-\frac{7}{2}}), \\ g_2'(s_0) &= F\left(\frac{5}{4}, \frac{5}{4}; \frac{3}{2}; -s_0^2\right) - \frac{25}{12}s_0^2 F\left(\frac{9}{4}, \frac{9}{4}; \frac{5}{2}; -s_0^2\right) \\ &= -\frac{3\sqrt{\pi}}{2\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})} s_0^{-\frac{5}{2}} \log(s_0) + \frac{3\sqrt{\pi}(\gamma + F(\frac{1}{4}) + \frac{8}{3})}{2\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})} s_0^{-\frac{5}{2}} + O(|s_0|^{-\frac{7}{2}}). \end{aligned}$$

Therefore we obtain the following estimates

$$\begin{aligned} |g_1(s)| &\lesssim \langle s \rangle^{-\frac{3}{2}} \langle \log \langle s \rangle \rangle, \quad |g_2(s)| \lesssim \langle s \rangle^{-\frac{3}{2}} \langle \log \langle s \rangle \rangle, \\ |g_1'(s_0)| &\lesssim \langle s_0 \rangle^{-\frac{5}{2}} \langle \log \langle s_0 \rangle \rangle, \quad |g_2'(s_0)| \lesssim \langle s_0 \rangle^{-\frac{5}{2}} \langle \log \langle s_0 \rangle \rangle, \end{aligned}$$

and as a result

$$\begin{aligned} |C_1(k, \eta)| &\lesssim \langle s_0 \rangle^{\frac{3}{2}} \langle \log \langle s_0 \rangle \rangle \left(\left| \hat{\psi}^0(k, \eta) \right| + \frac{|\hat{T}^0(k, \eta)|}{\langle s_0 \rangle |k|} \right), \\ |C_2(k, \eta)| &\lesssim \langle s_0 \rangle^{\frac{3}{2}} \langle \log \langle s_0 \rangle \rangle \left(\left| \hat{\psi}^0(k, \eta) \right| + \frac{|\hat{T}^0(k, \eta)|}{\langle s_0 \rangle |k|} \right). \end{aligned}$$

Therefore we have

$$\begin{aligned} \left| \hat{\phi}(t; k, \eta) \right| &= |C_1(k, \eta)g_1(s) + C_2(k, \eta)g_2(s)| \\ &\lesssim \langle s \rangle^{-\frac{3}{2}} \langle s_0 \rangle^{\frac{3}{2}} \langle \log \langle s \rangle \rangle \langle \log \langle s_0 \rangle \rangle \left(\left| \hat{\psi}^0(k, \eta) \right| + \frac{|\hat{T}^0(k, \eta)|}{\langle s_0 \rangle |k|} \right), \end{aligned}$$

from which the estimates of $|\hat{v}^x(t; k, \eta)|$, $|\hat{v}^y(t; k, \eta)|$ and $|\hat{\tau}(t; k, \eta)|$ follow. Then the decay rates of v^x, v^y, T can be obtained similarly as in the proof of Lemma 7. So we only sketch it. Notice that for any $a \geq \frac{1}{2}$, the function $h(x) = \frac{\langle x \rangle^a}{\langle \log \langle x \rangle \rangle}$ is increasing for all $x \geq 0$. So when $|s| \leq \frac{1}{2}|t|$ (implying $|s_0| \geq \frac{1}{2}|t|$), we have

$$\begin{aligned} \langle s \rangle^{-a} \langle s_0 \rangle^{\frac{3}{2}} \langle \log \langle s \rangle \rangle \langle \log \langle s_0 \rangle \rangle &\leq \langle s_0 \rangle^{\frac{3}{2}} \langle \log \langle s_0 \rangle \rangle \leq \frac{h(s_0)}{h(\frac{1}{2}t)} \langle s_0 \rangle^{\frac{3}{2}} \langle \log \langle s_0 \rangle \rangle \\ &\lesssim \langle t \rangle^{-a} \langle \log \langle t \rangle \rangle \langle s_0 \rangle^{\frac{3}{2}+a}. \end{aligned}$$

On the other hand, when $|s| \geq \frac{1}{2}|t|$, we have

$$\langle s \rangle^{-a} \langle s_0 \rangle^{\frac{3}{2}} \langle \log \langle s \rangle \rangle \langle \log \langle s_0 \rangle \rangle \lesssim \langle t \rangle^{-a} \langle \log \langle t \rangle \rangle \langle s_0 \rangle^{\frac{3}{2}+a},$$

since $\langle \log \langle s_0 \rangle \rangle \leq \langle s_0 \rangle^a$. So similar to the proof of Lemma 7, we get

$$\begin{aligned} \|P_{\neq 0} v^x\|_{L^2} &\lesssim \langle t \rangle^{-\frac{1}{2}} \langle \log \langle t \rangle \rangle \left(\|\psi^0\|_{H_x^1 H_y^2} + \|T^0\|_{L_x^2 H_y^1} \right), \\ \|v^y\|_{L^2} &\lesssim \langle t \rangle^{-\frac{3}{2}} \langle \log \langle t \rangle \rangle \left(\|\psi^0\|_{H_x^1 H_y^3} + \|T^0\|_{L_x^2 H_y^2} \right). \end{aligned}$$

and

$$\|P_{\neq 0} T\|_{L^2} \lesssim \langle t \rangle^{-\frac{1}{2}} \langle \log \langle t \rangle \rangle \left(\|\psi^0\|_{H_x^1 H_y^2} + \|T^0\|_{L_x^2 H_y^1} \right).$$

4.3 The case $B^2 = 0$

When $B^2 = 0$, that is, $\beta = 0$, then by (13), we get

$$\begin{aligned} (\partial_t + Ry\partial_x) \Delta \psi &= -\partial_x \left(\frac{\rho}{\rho_0} \right) g, \\ (\partial_t + Ry\partial_x) \left(\frac{\rho}{\rho_0} \right) &= 0. \end{aligned}$$

For convenience, we let $R = 1$. Again, we define

$$\begin{aligned} f(t; z, y) &= \omega(t; z + ty, y) = \omega(t; x, y), \\ \phi(t; z, y) &= \psi(t; z + ty, y) = \psi(t; x, y), \\ \tau(t; z, y) &= \frac{\rho}{\rho_0}(t; z + ty, y) = \frac{\rho}{\rho_0}(t; x, y). \end{aligned}$$

Then

$$\partial_t f(t; z, y) = g \partial_z \tau(t; z, y), \quad \partial_t \tau(t; z, y) = 0.$$

So

$$\begin{aligned} \hat{\tau}(t; k, \eta) &= \hat{\tau}(0; k, \eta), \\ \hat{f}(t; k, \eta) &= \hat{f}(0; k, \eta) + tikg\hat{\tau}(0; k, \eta) = \hat{\omega}^0(k, \eta) + tikg\hat{\rho}^0(k, \eta), \end{aligned}$$

where $\omega(0; x, y) = \omega^0(x, y)$, $\frac{\rho}{\rho_0}(0; x, y) = \rho^0(x, y)$. Thus by (21), we get

$$\begin{aligned} \left| \hat{\phi}(t; k, \eta) \right| &= \frac{1}{k^2(1+s^2)} \left| \hat{f}(t; \eta, k) \right| \\ &\lesssim \langle s \rangle^{-2} \langle s_0 \rangle^2 \left| \hat{\psi}^0(k, \eta) \right| + |t| \frac{1}{|k|} \langle s \rangle^{-2} \left| \hat{\rho}^0(k, \eta) \right|. \end{aligned}$$

So

$$\begin{aligned} |\hat{v}^x(t; k, \eta)| &\lesssim \langle s \rangle^{-1} \langle s_0 \rangle^2 |k| \left| \hat{\psi}^0(k, \eta) \right| + |t| \langle s \rangle^{-1} \left| \hat{\rho}^0(k, \eta) \right|, \\ |\hat{v}^y(t; k, \eta)| &\lesssim \langle s \rangle^{-2} \langle s_0 \rangle^2 |k| \left| \hat{\psi}^0(k, \eta) \right| + |t| \langle s \rangle^{-2} \left| \hat{\rho}^0(k, \eta) \right|. \end{aligned}$$

By Lemma 7, we get

$$\begin{aligned} \|P_{\neq 0} v^x\|_{L^2} &\lesssim \|\rho^0\|_{L_x^2 H_y^1} + \langle t \rangle^{-1} \|\psi^0\|_{H_x^1 H_y^3}, \\ \|v^y\|_{L^2} &\lesssim \langle t \rangle^{-1} \|\rho^0\|_{L_x^2 H_y^2} + \langle t \rangle^{-2} \|\psi^0\|_{H_x^1 H_y^4}. \end{aligned}$$

Also, $\left\| \frac{\rho}{\rho_0} \right\|_{L^2}(t) = \|\rho^0\|$. So when $\rho^0 \neq 0$, there is no decay for $\frac{\rho}{\rho_0}$ and $P_{\neq 0} v^x$. When $\rho^0 = 0$, we get

$$\|P_{\neq 0} v^x\|_{L^2} \lesssim \langle t \rangle^{-1} \|\psi^0\|_{H_x^1 H_y^3}, \quad \|v^y\|_{L^2} \lesssim \langle t \rangle^{-2} \|\psi^0\|_{H_x^1 H_y^4},$$

which exactly recovers the linear decay results in Lin and Zeng (2011) for the homogeneous fluids.

We note that for $B > 0$ small, the decay rates for $\|P_{\neq 0} v^x\|_{L^2}$ and $\|v^y\|_{L^2}$ are $t^{-\frac{1}{2}+\nu}$ and $t^{-\frac{3}{2}+\nu}$ respectively even when $\rho^0 = 0$. So if $B \rightarrow 0+$ ($\nu \rightarrow \frac{1}{2}-$), surprisingly the decay rates are almost one order slower than the case of homogeneous fluids ($B = 0$). This apparent gap is due to the vanishing of the coefficient of $\langle s \rangle^{-\frac{3}{2}+\nu}$ terms in the asymptotics of hypergeometric functions (34)-(37).

4.4 The case $B^2 = \infty$

When there is no shear, i.e. $R = 0$, $B^2 = \infty$, the equations (12-13) become

$$\partial_t \Delta \psi = -\partial_x \left(\frac{\rho}{\rho_0} \right) g, \quad \partial_t \left(\frac{\rho}{\rho_0} \right) = \beta \partial_x \psi.$$

Denote $T = \frac{\rho}{\beta \rho_0(y)}$, then

$$\Delta \psi_t = -\partial_x T \beta g, \tag{40}$$

$$\partial_t T = \partial_x \psi. \tag{41}$$

4.4.1 The L^2 Stability

Multiplying (40) by ψ and then integrating by parts with (41), we get the following invariant

$$\frac{d}{dt} \left(\beta g \int \int T^2 dx dy + \int \int |\nabla \psi|^2 dx dy \right) = 0.$$

This shows that in the L^2 norm, the perturbations of velocity and density are Liapunov stable but do not decay. However, below we show that their L^∞ norms decay due to the dispersive effects.

4.4.2 The L^∞ Decay

First, we solve (40)-(41) by Fourier transforms. Denote $N^2 = \beta g$ to be the squared Brunt-Väisälä frequency. By Fourier transform $(x, y) \rightarrow (k, \eta)$,

$$((i\eta)^2 + (ik)^2) \hat{\psi}_t = -(ik) N^2 \hat{T}, \quad (42)$$

$$\hat{T}_t = (ik) \hat{\psi}. \quad (43)$$

Combining (42)-(43), we get

$$\frac{d^2}{dt^2} \hat{\psi} = -\lambda^2 \hat{\psi},$$

where $\lambda^2(k, \eta) = \frac{k^2 N^2}{k^2 + \eta^2}$. For $k \neq 0$, its solutions are

$$\hat{\psi}(t) = C_1 e^{i\lambda t} + C_2 e^{-i\lambda t}.$$

By initial conditions,

$$\hat{\psi}(0) = C_1 + C_2 = \hat{\psi}^0, \quad \hat{\psi}'(0) = i\lambda(C_1 - C_2) = \frac{i\lambda^2}{k} \hat{T}^0,$$

thus

$$C_{1,2} = \frac{1}{2} \left(\hat{\psi}^0 \pm \frac{\lambda}{k} \hat{T}^0 \right).$$

By (43),

$$\hat{T} = -\frac{ik}{\lambda^2} \hat{\psi}_t = \frac{k}{\lambda} (C_1 e^{i\lambda t} - C_2 e^{-i\lambda t}).$$

To prove the L^∞ decay of solutions, we need two lemmas.

Lemma 8 (*Van der Corput*) *Let $h(x)$ be either convex or concave on $[a, b]$ with $-\infty \leq a < b \leq \infty$. Then*

$$\left| \int_b^a e^{ih(\eta)} d\eta \right| \leq 2 \left(\min_{[a,b]} |h'| \right)^{-1}, \quad \left| \int_b^a e^{ih(\eta)} d\eta \right| \leq 4 \left(\min_{[a,b]} |h''| \right)^{-\frac{1}{2}}. \quad (44)$$

Lemma 9 For $\lambda(k, \eta) = \frac{|k|N}{\sqrt{k^2 + \eta^2}}$ and n sufficiently large,

$$\left| \int_{-n}^n e^{i(\lambda t + \eta y)} d\eta \right| \lesssim |k|^{\frac{3}{2}} |Nt|^{-\frac{1}{3}} + |Nt|^{-\frac{1}{2}} |k|^{-\frac{1}{2}} n^{\frac{3}{2}}.$$

Proof. We can assume $N = 1$ without loss of generality. Notice that

$$\begin{aligned} \lambda(\eta) &= \frac{1}{\sqrt{1 + \left(\frac{\eta}{k}\right)^2}} = \left\langle \frac{\eta}{k} \right\rangle^{-1}, \\ \lambda'(\eta) &= -\frac{\eta}{k^2} \left\langle \frac{\eta}{k} \right\rangle^{-3}, \\ \lambda''(\eta) &= \frac{2\eta^2 - k^2}{k^4} \left\langle \frac{\eta}{k} \right\rangle^{-5}, \end{aligned}$$

and $\lambda(\eta)$ has two inflection point, $\eta_{1,2} = \pm \frac{\sqrt{2}}{2}k$. Let $n > \frac{\sqrt{2}}{2}|k|$. Choose $\epsilon > 0$ so small that all the Taylor's expansion below are valid in $(\eta_i - \epsilon, \eta_i + \epsilon)$, $i = 1, 2$. Define

$$S_1 = (-n, \eta_1 - \epsilon) \cup (\eta_1 + \epsilon, \eta_2 - \epsilon) \cup (\eta_2 + \epsilon, n).$$

By (44), we have

$$\begin{aligned} \left| \int_{S_1} e^{i(\lambda t + \eta y)} d\eta \right| &\leq 4 \left(\min_{[a,b]} |t| |\lambda''| \right)^{-\frac{1}{2}} \\ &= 4|t|^{-\frac{1}{2}} \left(\frac{2n^2 - k^2}{k^4} \left\langle \frac{n}{k} \right\rangle^{-5} \right)^{-\frac{1}{2}} \\ &\lesssim |k|^{-\frac{1}{2}} |t|^{-\frac{1}{2}} n^{\frac{3}{2}}, \end{aligned}$$

provided $n = n(\epsilon)$ is sufficiently large. For large t , we can divide $(\eta_1 - \epsilon, \eta_1 + \epsilon) = \left\{ |t|^{-\frac{1}{3}} < |\eta - \eta_1| < \epsilon \right\} \cup \left\{ |\eta - \eta_1| \leq |t|^{-\frac{1}{3}} \right\} = S_2 \cup S_3$, so that

$$\left| \int_{\eta_1 - \epsilon}^{\eta_1 + \epsilon} e^{i(\lambda t + \eta y)} d\eta \right| \leq 4|t|^{-\frac{1}{2}} \left(\min_{S_2} |\lambda''| \right)^{-\frac{1}{2}} + 2|t|^{-\frac{1}{3}}.$$

For $\eta \in S_2$, we have

$$\begin{aligned} |\lambda''(\eta)| &= \frac{|2\eta^2 - k^2|}{k^4} \left\langle \frac{\eta}{k} \right\rangle^{-5} \\ &= \frac{2|\eta - \eta_1||\eta - \eta_2|}{k^4} \left\langle \frac{\eta}{k} \right\rangle^{-5} \\ &> \frac{2|\eta - \eta_2|}{k^4} \left\langle \frac{\eta}{k} \right\rangle^{-5} |t|^{-\frac{1}{3}} \\ &\gtrsim |k|^{-3} |t|^{-\frac{1}{3}}. \end{aligned}$$

Therefore

$$\left| \int_{\eta_1 - \epsilon}^{\eta_1 + \epsilon} e^{i(\lambda t + \eta y)} d\eta \right| \lesssim 4|t|^{-\frac{1}{2}} \left(|k|^{-3} |t|^{-\frac{1}{3}} \right)^{-\frac{1}{2}} + 2|t|^{-\frac{1}{3}} \lesssim |k|^{\frac{3}{2}} |t|^{-\frac{1}{3}}.$$

Applying similar estimates to $(\eta_2 - \epsilon, \eta_2 + \epsilon)$ will complete the proof of this lemma. ■

Now we prove the L^∞ of the solutions of (40)-(41). By Fourier inverse transform formula,

$$\begin{aligned} P_{\neq 0}\psi(t; x, y) &= \frac{1}{2\pi} \sum_{k \neq 0} \left(e^{ikx} \int_{-\infty}^{\infty} \hat{\psi}(t) e^{i\eta y} d\eta \right) \\ &= \frac{1}{2\pi} \sum_{k \neq 0} \left(e^{ikx} \int_{-\infty}^{\infty} (C_1(k, \eta) e^{i\lambda t} + C_2(k, \eta) e^{-i\lambda t}) e^{i\eta y} d\eta \right) \end{aligned}$$

where

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} C_1(k, \eta) e^{i\lambda t} e^{i\eta y} d\eta \right| \\ &\leq \frac{1}{2} \left| \int_{-\infty}^{\infty} \hat{\psi}^0(k, \eta) e^{i\lambda t} e^{i\eta y} d\eta \right| + \frac{1}{2|k|} \left| \int_{-\infty}^{\infty} \lambda \hat{T}^0(k, \eta) e^{i\lambda t} e^{i\eta y} d\eta \right|. \end{aligned}$$

Define

$$\begin{aligned} I(y) &= \int_{-n}^n e^{i\lambda(k, \eta)t} \hat{\psi}^0(k, \eta) e^{i\eta y} d\eta \\ &= \left(e^{i\lambda(k, \eta)t} \chi_{[-n, n]} \hat{\psi}^0(k, \eta) \right)^\vee(y) \\ &= \left(e^{i\lambda(k, \eta)t} \chi_{[-n, n]} \right)^\vee * \hat{\psi}^0(k, y), \end{aligned}$$

then

$$\begin{aligned} \|I(y)\|_{L^\infty} &\leq \left\| \left(e^{i\lambda(k, \eta)t} \chi_{[-n, n]} \right)^\vee \right\|_{L_y^\infty} \|\hat{\psi}^0(k, \cdot)\|_{L_y^1} \\ &\leq \left\| \int_{-n}^n e^{i\lambda(k, \eta)t} e^{i\eta y} d\eta \right\|_{L_y^\infty} \|\hat{\psi}^0(k, \cdot)\|_{L_y^1}. \end{aligned}$$

Here, $^\vee$ stands for the inverse Fourier transform. By lemma 9, we have

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} \hat{\psi}^0(k, \eta) e^{i\lambda t} e^{i\eta y} d\eta \right| \\ &\leq \int_{|\eta| > n} |\hat{\psi}^0(k, \eta)| d\eta + |I(y)| \\ &\lesssim \left(\int_{|\eta| > n} \langle \eta \rangle^{-2\alpha} d\eta \right)^{\frac{1}{2}} \|\hat{\psi}^0(k, \cdot)\|_{H_y^\alpha} + \left(|k|^{\frac{3}{2}} |Nt|^{-\frac{1}{3}} + |k|^{-\frac{1}{2}} |Nt|^{-\frac{1}{2}} n^{\frac{3}{2}} \right) \|\hat{\psi}^0(k, \cdot)\|_{L_y^1} \\ &\lesssim \left(n^{-\alpha + \frac{1}{2}} + |k|^{\frac{3}{2}} |t|^{-\frac{1}{3}} + |k|^{-\frac{1}{2}} |t|^{-\frac{1}{2}} n^{\frac{3}{2}} \right) \left(\|\hat{\psi}^0(k, \cdot)\|_{H_y^\alpha} + \|\hat{\psi}^0(k, \cdot)\|_{L_y^1} \right). \end{aligned}$$

Choose $n = |t|^{\frac{1}{2\alpha+2}}$, for $\alpha \in (\frac{1}{2}, \frac{7}{2}]$, we have

$$\left| \int_{-\infty}^{\infty} \hat{\psi}^0(k, \eta) e^{i\lambda t} e^{i\eta y} d\eta \right| \lesssim |k|^{\frac{3}{2}} |t|^{-\frac{2\alpha-1}{4\alpha+4}} \left(\|\hat{\psi}^0\|_{H_y^\alpha} + \|\hat{\psi}^0\|_{L_y^1} \right).$$

If the initial condition is smooth enough, then

$$\left| \int_{-\infty}^{\infty} \hat{\psi}^0(k, \eta) e^{i\lambda t} e^{i\eta y} d\eta \right| \lesssim |k|^{\frac{3}{2}} |t|^{-\frac{1}{3}} \left(\|\hat{\psi}^0\|_{H_y^{7/2}} + \|\hat{\psi}^0\|_{L_y^1} \right).$$

Similarly,

$$\left| \int_{-\infty}^{\infty} \lambda \hat{T}^0(k, \eta) e^{i\lambda t} e^{i\eta y} d\eta \right| \lesssim N |k|^{\frac{3}{2}} |t|^{-\frac{1}{3}} \left(\|\hat{T}^0\|_{H_y^{7/2}} + \|\hat{T}^0\|_{L_y^1} \right).$$

Therefore, we have

$$\|P_{\neq 0} \hat{\psi}(t; k, \cdot)\|_{L_y^\infty} \lesssim |t|^{-\frac{1}{3}} \left(|k|^{\frac{3}{2}} \|\hat{\psi}^0\|_{H_y^{7/2}} + |k|^{\frac{3}{2}} \|\hat{\psi}^0\|_{L_y^1} + |k|^{\frac{1}{2}} \|\hat{T}^0\|_{H_y^{7/2}} + |k|^{\frac{1}{2}} \|\hat{T}^0\|_{L_y^1} \right).$$

So the decay in $L_x^2 L_y^\infty$ is obtained:

$$\begin{aligned} \|P_{\neq 0} \psi\|_{L_x^2 L_y^\infty} &\lesssim |t|^{-\frac{1}{3}} \left(\|\psi^0\|_{H_x^{3/2} H_y^{7/2}} + \|\psi^0\|_{H_x^{3/2} L_y^1} + \|T^0\|_{H_x^{1/2} H_y^{7/2}} + \|T^0\|_{H_x^{1/2} L_y^1} \right), \\ \|P_{\neq 0} v^x\|_{L_x^2 L_y^\infty} &\lesssim |t|^{-\frac{1}{3}} \left(\|\psi^0\|_{H_x^{3/2} H_y^{9/2}} + \|\psi^0\|_{H_x^{3/2} W_y^{1,1}} + \|T^0\|_{H_x^{1/2} H_y^{9/2}} + \|T^0\|_{H_x^{1/2} W_y^{1,1}} \right), \\ \|v^y\|_{L_x^2 L_y^\infty} &\lesssim |t|^{-\frac{1}{3}} \left(\|\psi^0\|_{H_x^{5/2} H_y^{7/2}} + \|\psi^0\|_{H_x^{5/2} L_y^1} + \|T^0\|_{H_x^{3/2} H_y^{7/2}} + \|T^0\|_{H_x^{3/2} L_y^1} \right). \end{aligned}$$

Similarly, for the density we have

$$\|P_{\neq 0} T\|_{L_x^2 L_y^\infty} \lesssim |t|^{-\frac{1}{3}} \left(\|\psi^0\|_{H_x^{5/2} H_y^{9/2}} + \|\psi^0\|_{H_x^{5/2} W_y^{1,1}} + \|T^0\|_{H_x^{3/2} H_y^{7/2}} + \|T^0\|_{H_x^{3/2} L_y^1} \right).$$

5 Decay estimates for the full Euler equation

In this section, we prove the decay estimates in Theorem 2 for the linearized system of the full Euler equation. The proof is very similar to the Boussinesq case. So we only sketch it.

5.1 The case $0 < B^2 < \infty$

For each $B^2 > 0$, we can find similar bounds for

$$\hat{\chi} = C_3(k, \eta) g_3(s) + C_4(k, \eta) g_4(s)$$

as in the Boussinesq case. For $B^2 > 0$ and $B^2 \neq \frac{1}{4}$, the asymptotics of g_3, g_4 at $s = \infty$ are

$$\begin{aligned}
g_3(s) &= \left(\frac{1}{2} - \frac{i\kappa s}{2}\right)^{-1-\beta_1} \left[\frac{\Gamma(2-\beta_1)\Gamma(-2\nu)}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{1}{2}-\beta_1-\nu)} \left(-\frac{i\kappa s}{2}\right)^{-\frac{1}{2}+\beta_1-\nu} \right. \\
&\quad \left. + \frac{\Gamma(2-\beta_1)\Gamma(2\nu)}{\Gamma(\frac{3}{2}+\nu)\Gamma(\frac{1}{2}-\beta_1+\nu)} \left(-\frac{i\kappa s}{2}\right)^{-\frac{1}{2}+\beta_1+\nu} + O\left(|s|^{-\frac{3}{2}+Re(\nu)}\right) \right], \\
g_4(s) &= \left(\frac{1}{2} + \frac{i\kappa s}{2}\right)^{-1+\beta_1} \left[\frac{\Gamma(\beta_1)\Gamma(-2\nu)}{\Gamma(-\frac{1}{2}-\nu)\Gamma(\frac{1}{2}+\beta_1-\nu)} \left(-\frac{i\kappa s}{2}\right)^{-\frac{1}{2}-\beta_1-\nu} \right. \\
&\quad \left. + \frac{\Gamma(\beta_1)\Gamma(2\nu)}{\Gamma(-\frac{1}{2}+\nu)\Gamma(\frac{1}{2}+\beta_1+\nu)} \left(-\frac{i\kappa s}{2}\right)^{-\frac{1}{2}-\beta_1+\nu} + O\left(|s|^{-\frac{3}{2}+Re(\nu)}\right) \right], \\
g'_3(s) &= \left(\frac{i\kappa}{2}\right) \left(\frac{1}{2} - \frac{i\kappa s}{2}\right)^{-\beta_1} \left[\frac{(\frac{3}{2}+\nu)\Gamma(2-\beta_1)\Gamma(-2\nu)}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{1}{2}-\beta_1-\nu)} \left(-\frac{i\kappa s}{2}\right)^{-\frac{5}{2}+\beta_1-\nu} \right. \\
&\quad \left. + \frac{(\frac{3}{2}-\nu)\Gamma(2-\beta_1)\Gamma(2\nu)}{\Gamma(\frac{3}{2}+\nu)\Gamma(\frac{1}{2}-\beta_1+\nu)} \left(-\frac{i\kappa s}{2}\right)^{-\frac{5}{2}+\beta_1+\nu} + O\left(|s|^{-\frac{7}{2}+Re(\nu)}\right) \right], \\
g'_4(s) &= \left(\frac{i\kappa}{2}\right) \left(\frac{1}{2} + \frac{i\kappa s}{2}\right)^{\beta_1} \left[\frac{(-\frac{3}{2}-\nu)\Gamma(2-\beta_1)\Gamma(-2\nu)}{\Gamma(-\frac{1}{2}-\nu)\Gamma(\frac{1}{2}+\beta_1-\nu)} \left(\frac{i\kappa s}{2}\right)^{-\frac{5}{2}-\beta_1-\nu} \right. \\
&\quad \left. + \frac{(-\frac{3}{2}+\nu)\Gamma(2-\beta_1)\Gamma(2\nu)}{\Gamma(-\frac{1}{2}+\nu)\Gamma(\frac{1}{2}+\beta_1+\nu)} \left(\frac{i\kappa s}{2}\right)^{-\frac{5}{2}-\beta_1+\nu} + O\left(|s|^{-\frac{7}{2}+Re(\nu)}\right) \right].
\end{aligned}$$

For $B^2 = \frac{1}{4}$, the expansions at $s = \infty$ are

$$\begin{aligned}
g_3(s) &= \left(\frac{1}{2} - \frac{i\kappa s}{2}\right)^{-\beta_1} \left[\frac{2\Gamma(2-\beta)}{\sqrt{\pi}\Gamma(\frac{1}{2}-\beta)} \left(-\frac{i\kappa s}{2}\right)^{-\frac{3}{2}+\beta_1} \log\left(-\frac{i\kappa s}{2}\right) + O\left(|s|^{-\frac{3}{2}+\beta_1}\right) \right], \\
g_4(s) &= \left(\frac{1}{2} + \frac{i\kappa s}{2}\right)^{\beta_1} \left[\frac{\Gamma(\beta)}{2\sqrt{\pi}\Gamma(\frac{1}{2}+\beta)} \left(-\frac{i\kappa s}{2}\right)^{-\frac{3}{2}-\beta_1} \log\left(-\frac{i\kappa s}{2}\right) + O\left(|s|^{-\frac{3}{2}-\beta_1}\right) \right], \\
g'_3(s) &= \left(\frac{i\kappa}{2}\right) \left(\frac{1}{2} - \frac{i\kappa s}{2}\right)^{-\beta_1} \left[\frac{3\Gamma(2-\beta)}{\sqrt{\pi}\Gamma(\frac{1}{2}-\beta)} \left(-\frac{i\kappa s}{2}\right)^{-\frac{5}{2}+\beta_1} \log\left(-\frac{i\kappa s}{2}\right) + O\left(|s|^{-\frac{5}{2}+\beta_1}\right) \right], \\
g'_4(s) &= \left(\frac{i\kappa}{2}\right) \left(\frac{1}{2} + \frac{i\kappa s}{2}\right)^{\beta_1} \left[\frac{3\Gamma(\beta)}{4\sqrt{\pi}\Gamma(\frac{1}{2}+\beta)} \left(-\frac{i\kappa s}{2}\right)^{-\frac{5}{2}-\beta_1} \log\left(-\frac{i\kappa s}{2}\right) + O\left(|s|^{-\frac{5}{2}-\beta_1}\right) \right].
\end{aligned}$$

Thus, we have the same bounds on $\hat{\chi}$, that is,

$$|\hat{\chi}(t; k, \eta)| \lesssim \langle s \rangle^{-\frac{3}{2}+\nu} \langle s_0 \rangle^{\frac{3}{2}+\nu} \left(\left| \hat{\Psi}^0(k, \eta) \right| + \frac{|\hat{\Upsilon}^0(k, \eta)|}{\langle s_0 \rangle |k|} \right),$$

when $0 < B^2 < \frac{1}{4}$;

$$|\hat{\chi}(t; k, \eta)| \lesssim \langle s \rangle^{-\frac{3}{2}} \langle s_0 \rangle^{\frac{3}{2}} \left(\left| \hat{\Psi}^0(k, \eta) \right| + \frac{|\hat{\Upsilon}^0(k, \eta)|}{\langle s_0 \rangle |k|} \right),$$

when $B^2 > \frac{1}{4}$, and

$$|\hat{\chi}(t; k, \eta)| \lesssim \langle s \rangle^{-\frac{3}{2}} \langle s_0 \rangle^{\frac{3}{2}} \langle \log \langle s \rangle \rangle \langle \log \langle s_0 \rangle \rangle \left(\left| \hat{\Psi}^0(k, \eta) \right| + \frac{|\hat{\Upsilon}^0(k, \eta)|}{\langle s_0 \rangle |k|} \right),$$

when $B^2 = \frac{1}{4}$.

Since

$$\begin{aligned} e^{-\frac{1}{2}\beta y} v^x(t; x, y) &= e^{-\frac{1}{2}\beta y} (-\partial_y \psi(t; x, y)) = e^{-\frac{1}{2}\beta y} (-\partial_y + t\partial_z) \phi(t; z, y) \\ &= (-\partial_y + t\partial_z) \left(e^{-\frac{1}{2}\beta y} \phi(t; z, y) \right) - \frac{1}{2} \beta e^{-\frac{1}{2}\beta y} \phi(t; z, y) \\ &= \left(-\partial_y + t\partial_z - \frac{1}{2} \beta \right) \chi(t; x, y), \\ e^{-\frac{1}{2}\beta y} v^y(t; x, y) &= e^{-\frac{1}{2}\beta y} \partial_x \psi(t; x, y) = \partial_x e^{-\frac{1}{2}\beta y} \phi(t; x - ty, y) = \partial_z \chi(t; z, y), \end{aligned}$$

the decay estimates for $e^{-\frac{1}{2}\beta y} v^x$ and $e^{-\frac{1}{2}\beta y} v^y$ (in Theorem 2 (i)-(iii)) can be proved as in the Boussinesq case. Decay of density can be obtained similarly.

5.2 The case $B^2 = 0$

When $B^2 = 0$, i.e., $\beta = 0$, the linearized equations are exactly the same as the Boussinesq case. Thus all the estimates are the same.

5.3 The case $B^2 = \infty$

When there is no shear, i.e. $R = 0$, the equations (7-8) are

$$\begin{aligned} -\beta \partial_t \partial_y \psi + \partial_t \Delta \psi &= -\partial_x \left(\frac{\rho}{\rho_0} \right) g, \\ \partial_t \left(\frac{\rho}{\rho_0} \right) &= \beta \partial_x \psi. \end{aligned}$$

Likewise, denote $T = \frac{\rho}{\beta \rho_0(y)}$, then the equation reads

$$(-\beta \partial_y + \Delta) \psi_t = -\partial_x T \beta g, \quad (45)$$

$$\partial_t T = \partial_x \psi. \quad (46)$$

Let $\Psi = e^{-\frac{1}{2}\beta y} \psi$, $\Upsilon = e^{-\frac{1}{2}\beta y} T$, then the equations (45)-(46) become

$$\left(-\frac{1}{4} \beta^2 + \Delta \right) \Psi_t = -N^2 \partial_x \Upsilon, \quad \partial_t \Upsilon = \partial_x \Psi. \quad (47)$$

By Fourier transform $(x, y) \rightarrow (k, \eta)$, we have

$$\left(-\frac{1}{4}\beta^2 + (i\eta)^2 + (ik)^2\right) \hat{\Psi}_t = -(ik)N^2 \hat{\Upsilon}, \quad \hat{\Upsilon}_t = (ik)\hat{\Psi}.$$

So

$$\frac{d^2}{dt^2} \hat{\Psi} = -\lambda^2 \hat{\Psi},$$

where

$$\lambda^2 = \frac{k^2 N^2}{k^2 + \eta^2 + \frac{\beta^2}{4}}.$$

Its solutions are

$$\hat{\Psi}(t) = C_1 e^{i\lambda t} + C_2 e^{-i\lambda t},$$

where

$$C_{1,2} = \frac{1}{2} \left(\hat{\Psi}^0 \pm \frac{\lambda}{k} \hat{\Upsilon}^0 \right).$$

Similar to the Boussinesq case, we have the following conservation law for (47)

$$0 = \frac{d}{dt} \left(\int \int \left(\frac{1}{4} \beta^2 |\Psi|^2 + |\nabla \Psi|^2 + N^2 |\Upsilon|^2 \right) dx dy \right).$$

By integration by parts

$$\begin{aligned} & \int \int \left(\frac{1}{4} \beta^2 |\Psi|^2 + |\nabla \Psi|^2 + N^2 |\Upsilon|^2 \right) dx dy \\ &= \left\| e^{-\frac{1}{2}\beta y} v^x \right\|_{L^2}^2 + \left\| e^{-\frac{1}{2}\beta y} v^y \right\|_{L^2}^2 + \frac{g}{\beta} \left\| e^{-\frac{1}{2}\beta y} \frac{\rho}{\rho_0} \right\|_{L^2}^2. \end{aligned}$$

This show that there is no decay in the L^2 norm for $e^{-\frac{1}{2}\beta y} v$ and $e^{-\frac{1}{2}\beta y} \frac{\rho}{\rho_0}$. For the L^∞ decay, notice that

$$\lambda^2 = \frac{k^2 N^2}{k^2 + \eta^2 + \frac{\beta^2}{4}} = \frac{m^2 (\kappa N)^2}{m^2 + \eta^2}.$$

where $m = \sqrt{\frac{1}{4}\beta^2 + k^2}$, $\kappa = \frac{k}{m}$. By lemma 9 we have

$$\begin{aligned} \left| \int_{-n}^n e^{i(\lambda t + \eta y)} d\eta \right| &\lesssim |m|^{\frac{3}{2}} |\kappa N t|^{-\frac{1}{3}} + |\kappa N t|^{-\frac{1}{2}} |m|^{-\frac{1}{2}} n^{\frac{3}{2}} \\ &\simeq |k|^{\frac{3}{2}} |t|^{-\frac{1}{3}} + |t|^{-\frac{1}{2}} |k|^{-\frac{1}{2}} n^{\frac{3}{2}} \end{aligned}$$

since $\kappa \simeq 1$, $m \simeq k$. Accordingly, we have

$$\begin{aligned} \|e^{-\frac{1}{2}\beta y} P_{\neq 0} \psi\|_{L_x^2 L_y^\infty} &\lesssim |t|^{-\frac{1}{3}} \left(\|\Psi^0\|_{H_x^{3/2} H_y^{7/2}} + \|\Psi^0\|_{H_x^{3/2} L_y^1} + \|\Upsilon^0\|_{H_x^{1/2} H_y^{7/2}} + \|\Upsilon^0\|_{H_x^{1/2} L_y^1} \right), \\ \|e^{-\frac{1}{2}\beta y} P_{\neq 0} v^x\|_{L_x^2 L_y^\infty} &\lesssim |t|^{-\frac{1}{3}} \left(\|\Psi^0\|_{H_x^{3/2} H_y^{9/2}} + \|\Psi^0\|_{H_x^{3/2} W_y^{1,1}} + \|\Upsilon^0\|_{H_x^{1/2} H_y^{9/2}} + \|\Upsilon^0\|_{H_x^{1/2} W_y^{1,1}} \right), \\ \|e^{-\frac{1}{2}\beta y} v^y\|_{L_x^2 L_y^\infty} &\lesssim |t|^{-\frac{1}{3}} \left(\|\Psi^0\|_{H_x^{5/2} H_y^{7/2}} + \|\Psi^0\|_{H_x^{5/2} L_y^1} + \|\Upsilon^0\|_{H_x^{3/2} H_y^{7/2}} + \|\Upsilon^0\|_{H_x^{3/2} L_y^1} \right), \\ \|e^{-\frac{1}{2}\beta y} P_{\neq 0} T\|_{L_x^2 L_y^\infty} &\lesssim |t|^{-\frac{1}{3}} \left(\|\Psi^0\|_{H_x^{5/2} H_y^{9/2}} + \|\Psi^0\|_{H_x^{5/2} W_y^{1,1}} + \|\Upsilon^0\|_{H_x^{3/2} H_y^{7/2}} + \|\Upsilon^0\|_{H_x^{3/2} L_y^1} \right). \end{aligned}$$

Appendix

In this appendix, we prove the two lemmas about the hypergeometric functions.
First we prove Lemma 5.

Proof of Lemma 5. Define

$$\begin{aligned} G(z) &= F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) \\ &+ F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; -\frac{1}{2}; z\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; z\right) \\ &- F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; z\right). \end{aligned}$$

In order to obtain $G'(z)$, we calculate the derivative of each term by Lemma 3,

$$\begin{aligned}
& \frac{d}{dz} F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) \\
&= \frac{\left(\frac{3}{4} - \frac{\nu}{2}\right)\left(\frac{3}{4} + \frac{\nu}{2}\right)}{\frac{1}{2}} F\left(\frac{7}{4} - \frac{\nu}{2}, \frac{7}{4} + \frac{\nu}{2}; \frac{3}{2}; z\right) \\
&= 2\left(\frac{3}{4} - \frac{\nu}{2}\right)\left(\frac{3}{4} + \frac{\nu}{2}\right) F\left(\frac{7}{4} - \frac{\nu}{2}, \frac{7}{4} + \frac{\nu}{2}; \frac{3}{2}; z\right), \\
& \frac{d}{dz} F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) \\
&= \frac{1}{\left(\frac{1}{2}\right)(1-z)} \left[\left(\frac{1}{2} - \left(\frac{5}{4} - \frac{\nu}{2}\right)\right) \left(\frac{1}{2} - \left(\frac{5}{4} + \frac{\nu}{2}\right)\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; z\right) \right. \\
&\quad \left. + \frac{1}{2} \left(\frac{5}{4} - \frac{\nu}{2} + \frac{5}{4} + \frac{\nu}{2} - \frac{1}{2}\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) \right] \\
&= \frac{2}{1-z} \left[\left(-\frac{3}{4} + \frac{\nu}{2}\right) \left(-\frac{3}{4} - \frac{\nu}{2}\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; z\right) + F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) \right] \\
&= \frac{2}{1-z} \left[\left(\frac{9}{16} - \frac{\nu^2}{4}\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; z\right) + F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) \right], \\
& \frac{d}{dz} F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; -\frac{1}{2}; z\right) \\
&= \frac{1}{\left(-\frac{1}{2}\right)(1-z)} \left[\left(-\frac{1}{2} - \left(\frac{3}{4} - \frac{\nu}{2}\right)\right) \left(-\frac{1}{2} - \left(\frac{3}{4} + \frac{\nu}{2}\right)\right) F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) \right. \\
&\quad \left. + \left(-\frac{1}{2}\right) \left(\frac{3}{4} - \frac{\nu}{2} + \frac{3}{4} + \frac{\nu}{2} - \left(-\frac{1}{2}\right)\right) F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; -\frac{1}{2}; z\right) \right] \\
&= -\frac{2}{1-z} \left[\left(-\frac{5}{4} + \frac{\nu}{2}\right) \left(-\frac{5}{4} - \frac{\nu}{2}\right) F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) - F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; -\frac{1}{2}; z\right) \right] \\
&= -\frac{2}{1-z} \left[\left(\frac{25}{16} - \frac{\nu^2}{4}\right) F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) - F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; -\frac{1}{2}; z\right) \right], \\
& \frac{d}{dz} F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; z\right) \\
&= \frac{\left(\frac{5}{4} - \frac{\nu}{2}\right)\left(\frac{5}{4} + \frac{\nu}{2}\right)}{\frac{3}{2}} F\left(\frac{9}{4} - \frac{\nu}{2}, \frac{9}{4} + \frac{\nu}{2}; \frac{5}{2}; z\right) \\
&= \frac{2}{3} \left(\frac{5}{4} - \frac{\nu}{2}\right) \left(\frac{5}{4} + \frac{\nu}{2}\right) F\left(\frac{9}{4} - \frac{\nu}{2}, \frac{9}{4} + \frac{\nu}{2}; \frac{5}{2}; z\right).
\end{aligned}$$

Hence

$$\begin{aligned}
G'(z) &= \left[2 \left(\frac{3}{4} - \frac{\nu}{2} \right) \left(\frac{3}{4} + \frac{\nu}{2} \right) F \left(\frac{7}{4} - \frac{\nu}{2}, \frac{7}{4} + \frac{\nu}{2}; \frac{3}{2}; z \right) \right] F \left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{1}{2}; z \right) \\
&+ F \left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; z \right) \\
&\quad \times \frac{2}{1-z} \left[\left(\frac{9}{16} - \frac{\nu^2}{4} \right) F \left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; z \right) + F \left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{1}{2}; z \right) \right] \\
&- \frac{2}{1-z} \left[\left(\frac{25}{16} - \frac{\nu^2}{4} \right) F \left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; z \right) - F \left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; -\frac{1}{2}; z \right) \right] \\
&\quad \times F \left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; z \right) \\
&+ F \left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; -\frac{1}{2}; z \right) \left[\frac{2}{3} \left(\frac{5}{4} - \frac{\nu}{2} \right) \left(\frac{5}{4} + \frac{\nu}{2} \right) F \left(\frac{9}{4} - \frac{\nu}{2}, \frac{9}{4} + \frac{\nu}{2}; \frac{5}{2}; z \right) \right] \\
&- \left[2 \left(\frac{3}{4} - \frac{\nu}{2} \right) \left(\frac{3}{4} + \frac{\nu}{2} \right) F \left(\frac{7}{4} - \frac{\nu}{2}, \frac{7}{4} + \frac{\nu}{2}; \frac{3}{2}; z \right) \right] F \left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; z \right) \\
&- F \left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; z \right) \left[\frac{2}{3} \left(\frac{5}{4} - \frac{\nu}{2} \right) \left(\frac{5}{4} + \frac{\nu}{2} \right) F \left(\frac{9}{4} - \frac{\nu}{2}, \frac{9}{4} + \frac{\nu}{2}; \frac{5}{2}; z \right) \right].
\end{aligned}$$

Notice

$$\begin{aligned}
& \left[2 \left(\frac{3}{4} - \frac{\nu}{2} \right) \left(\frac{3}{4} + \frac{\nu}{2} \right) F \left(\frac{7}{4} - \frac{\nu}{2}, \frac{7}{4} + \frac{\nu}{2}; \frac{3}{2}; z \right) \right] F \left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{1}{2}; z \right) \\
+ & F \left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; -\frac{1}{2}; z \right) \left[\frac{2}{3} \left(\frac{5}{4} - \frac{\nu}{2} \right) \left(\frac{5}{4} + \frac{\nu}{2} \right) F \left(\frac{9}{4} - \frac{\nu}{2}, \frac{9}{4} + \frac{\nu}{2}; \frac{5}{2}; z \right) \right] \\
- & \left[2 \left(\frac{3}{4} - \frac{\nu}{2} \right) \left(\frac{3}{4} + \frac{\nu}{2} \right) F \left(\frac{7}{4} - \frac{\nu}{2}, \frac{7}{4} + \frac{\nu}{2}; \frac{3}{2}; z \right) \right] F \left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; z \right) \\
- & F \left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; z \right) \left[\frac{2}{3} \left(\frac{5}{4} - \frac{\nu}{2} \right) \left(\frac{5}{4} + \frac{\nu}{2} \right) F \left(\frac{9}{4} - \frac{\nu}{2}, \frac{9}{4} + \frac{\nu}{2}; \frac{5}{2}; z \right) \right] \\
= & 2 \left(\frac{3}{4} - \frac{\nu}{2} \right) \left(\frac{3}{4} + \frac{\nu}{2} \right) F \left(\frac{7}{4} - \frac{\nu}{2}, \frac{7}{4} + \frac{\nu}{2}; \frac{3}{2}; z \right) \\
& \times \left[F \left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{1}{2}; z \right) - F \left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; z \right) \right] \\
+ & \frac{2}{3} \left(\frac{5}{4} - \frac{\nu}{2} \right) \left(\frac{5}{4} + \frac{\nu}{2} \right) F \left(\frac{9}{4} - \frac{\nu}{2}, \frac{9}{4} + \frac{\nu}{2}; \frac{5}{2}; z \right) \\
& \times \left[F \left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; -\frac{1}{2}; z \right) - F \left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; z \right) \right] \\
= & 2 \left(\frac{3}{4} - \frac{\nu}{2} \right) \left(\frac{3}{4} + \frac{\nu}{2} \right) F \left(\frac{7}{4} - \frac{\nu}{2}, \frac{7}{4} + \frac{\nu}{2}; \frac{3}{2}; z \right) \\
& \times \frac{z}{\left(\frac{1}{2} \right) \left(\frac{3}{2} \right)} \left(\frac{5}{4} - \frac{\nu}{2} \right) \left(\frac{5}{4} + \frac{\nu}{2} \right) F \left(\frac{9}{4} - \frac{\nu}{2}, \frac{9}{4} + \frac{\nu}{2}; \frac{5}{2}; z \right) \\
+ & \frac{2}{3} \left(\frac{5}{4} - \frac{\nu}{2} \right) \left(\frac{5}{4} + \frac{\nu}{2} \right) F \left(\frac{9}{4} - \frac{\nu}{2}, \frac{9}{4} + \frac{\nu}{2}; \frac{5}{2}; z \right) \\
& \times \frac{z}{\left(-\frac{1}{2} \right) \left(\frac{1}{2} \right)} \left(\frac{3}{4} - \frac{\nu}{2} \right) \left(\frac{3}{4} + \frac{\nu}{2} \right) F \left(\frac{7}{4} - \frac{\nu}{2}, \frac{7}{4} + \frac{\nu}{2}; \frac{3}{2}; z \right) \\
= & 0,
\end{aligned}$$

we have

$$\begin{aligned}
G'(z) &= F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) \\
&\quad \times \frac{2}{1-z} \left[\left(\frac{9}{16} - \frac{\nu^2}{4}\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; z\right) + F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) \right] \\
&\quad - \frac{2}{1-z} \left[\left(\frac{25}{16} - \frac{\nu^2}{4}\right) F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) - F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; -\frac{1}{2}; z\right) \right] \\
&\quad \times F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; z\right) \\
&= \frac{2}{1-z} \left\{ \left[\left(\frac{9}{16} - \frac{\nu^2}{4}\right) - \left(\frac{25}{16} - \frac{\nu^2}{4}\right) \right] F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; z\right) \right. \\
&\quad + F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{1}{2}; z\right) \\
&\quad \left. + F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; -\frac{1}{2}; z\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; z\right) \right\} \\
&= \frac{2}{1-z} G(z).
\end{aligned}$$

Since

$$\begin{aligned}
G(0) &= F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; 0\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{1}{2}; 0\right) \\
&\quad + F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; -\frac{1}{2}; 0\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; 0\right) \\
&\quad - F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; 0\right) F\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; 0\right) \\
&= 1 + 1 - 1 = 1,
\end{aligned}$$

$G(z)$ is the solution to the ordinary differential equation

$$\begin{cases} G'(z) = \frac{2}{1-z} G(z), \\ G(0) = 1. \end{cases}$$

Therefore, $G(z) = \frac{1}{(1-z)^2}$. ■

Next, we prove Lemma 6.

Proof of Lemma 6. Define

$$H(\nu, \beta_1; z) = (1 - \beta_1 + 2z\beta_1) F_1(z) F_2(z) - z(1-z) (F_1(z) F_2'(z) - F_1'(z) F_2(z)).$$

One can always calculate its series expansion at $z = 1$ according to Gauss (Prop. 2.2.13) and Euler transform. We let computer to do this job. Around $z = 1$, we have

$$H(z) = 1 - \beta_1 + O(1-z)^2.$$

As for $z = \infty$, by Pfaff transform we can also obtain its expansion at infinity.

$$H(z) = 1 - \beta_1 + O\left(\frac{1}{z}\right)^7.$$

These means $\lim_{z \rightarrow 1} H(z) = \lim_{z \rightarrow \infty} H(z) = 1 - \beta_1$.

We first show that $H(1)$ and $H(\infty)$ both equal to a constant, and then show $H(z)$ is actually a constant for all $z \in \mathbb{C}$.

First we determine $H(1)$, which is not properly defined, so we are in fact looking for $\lim_{z \rightarrow 1} H(z)$. In order to do this, we use Euler transform (19) and equation (20) to find a series expansion of F_1 and F_2 near $z = 1$.

By Riemann, $F_1(z)$ near $z = 1$ can be locally expressed as

$$F_1(z) = (1 - z)^{\alpha_1} f_1(z) + (1 - z)^{\alpha_2} f_2(z),$$

where $f_1(z)$ and $f_2(z)$ are two analytic functions, and $f_1(z), f_2(z) \neq 0$. Since $a + b = \frac{1}{2} - \beta_1 - \nu + \frac{1}{2} - \beta_1 + \nu = 1 - 2\beta_1 < 2 - \beta_1 = c$, by equation (20) we know

$$F_1(1) = \frac{\Gamma(2 - \beta_1)\Gamma(1 + \beta_1)}{\Gamma\left(\frac{3}{2} - \nu\right)\Gamma\left(\frac{3}{2} + \nu\right)}$$

is a non-zero finite number, so we can assume $\alpha_1 = 0$. Moreover,

$$\begin{aligned} F_1'(1) &= \frac{\left(\frac{1}{2} - \beta_1 - \nu\right)\left(\frac{1}{2} - \beta_1 + \nu\right)}{2 - \beta_1} F\left(\frac{3}{2} - \beta_1 - \nu, \frac{3}{2} - \beta_1 + \nu; 3 - \beta_1; 1\right) \\ &= \frac{\left(\frac{1}{2} - \beta_1 - \nu\right)\left(\frac{1}{2} - \beta_1 + \nu\right)}{2 - \beta_1} \frac{\Gamma(3 - \beta_1)\Gamma(\beta_1)}{\Gamma\left(\frac{3}{2} - \nu\right)\Gamma\left(\frac{3}{2} + \nu\right)} \\ &= \left(\frac{1}{2} - \beta_1 - \nu\right)\left(\frac{1}{2} - \beta_1 + \nu\right) \frac{\Gamma(2 - \beta_1)\Gamma(\beta_1)}{\Gamma\left(\frac{3}{2} - \nu\right)\Gamma\left(\frac{3}{2} + \nu\right)}. \end{aligned}$$

So we have

$$f_1(z) = a_0 - a_1(z - 1) + O(1 - z)^2$$

where $a_0 = F_1(1), a_1 = F_1'(1)$. Since $\frac{5}{2} - \beta_1 - \nu + \frac{5}{2} - \beta_1 + \nu > 4 - \beta_1$,

$$\begin{aligned} F''(z) &= \frac{\left(\frac{1}{2} - \beta_1 - \nu\right)\left(\frac{1}{2} - \beta_1 + \nu\right)}{2 - \beta_1} \frac{\left(\frac{3}{2} - \beta_1 - \nu\right)\left(\frac{3}{2} - \beta_1 + \nu\right)}{3 - \beta_1} \\ &\quad \times F\left(\frac{5}{2} - \beta_1 - \nu, \frac{5}{2} - \beta_1 + \nu; 4 - \beta_1; z\right) \rightarrow \infty \end{aligned}$$

as $z \rightarrow 1$. By Euler transform (19), we have

$$F\left(\frac{5}{2} - \beta_1 - \nu, \frac{5}{2} - \beta_1 + \nu; 4 - \beta_1; z\right) = (1 - z)^{-1 + \beta_1} F\left(\frac{3}{2} - \nu, \frac{3}{2} + \nu; 4 - \beta_1; z\right).$$

Again by Gauss's (20), we know that

$$F\left(\frac{3}{2} - \nu, \frac{3}{2} + \nu; 4 - \beta_1; 1\right) = \frac{\Gamma(4 - \beta_1)\Gamma(1 - \beta_1)}{\Gamma\left(\frac{5}{2} - \beta_1 - \nu\right)\Gamma\left(\frac{5}{2} - \beta_1 + \nu\right)},$$

so

$$\begin{aligned}
F''(z) &= \frac{\left(\frac{1}{2} - \beta_1 - \nu\right) \left(\frac{1}{2} - \beta_1 + \nu\right) \left(\frac{3}{2} - \beta_1 - \nu\right) \left(\frac{3}{2} - \beta_1 + \nu\right)}{2 - \beta_1} \frac{3 - \beta_1}{3 - \beta_1} \\
&\quad \times \frac{\Gamma(4 - \beta_1)\Gamma(1 - \beta_1)}{\Gamma\left(\frac{5}{2} - \beta_1 - \nu\right)\Gamma\left(\frac{5}{2} - \beta_1 + \nu\right)} (1 - z)^{-1+\beta_1} + o(1 - z)^{-1+\beta_1} \\
&= \frac{\Gamma(2 - \beta_1)\Gamma(1 - \beta_1)}{\Gamma\left(\frac{1}{2} - \beta_1 - \nu\right)\Gamma\left(\frac{1}{2} - \beta_1 + \nu\right)} (1 - z)^{-1+\beta_1} + o(1 - z)^{-1+\beta_1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
F''(z) &= \frac{\partial^2}{\partial z^2} (f_1(z) + (1 - z)^{\alpha_2} f_2(z)) \\
&= O(1) + \alpha_2(\alpha_2 - 1)(1 - z)^{\alpha_2-2} f_2(z) + O(1 - z)^{\alpha_2-1} \\
&= \alpha_2(\alpha_2 - 1)(1 - z)^{\alpha_2-2} f_2(1) + o(1 - z)^{\alpha_2-2}.
\end{aligned}$$

Comparing the leading term in the right-hand sides of the last two equations, we know the index $\alpha_2 - 2 = -1 + \beta_1$, $\alpha_2 = 1 + \beta_1$, and the coefficient

$$\begin{aligned}
\frac{\Gamma(2 - \beta_1)\Gamma(1 - \beta_1)}{\Gamma\left(\frac{1}{2} - \beta_1 - \nu\right)\Gamma\left(\frac{1}{2} - \beta_1 + \nu\right)} &= \alpha_2(\alpha_2 - 1)f_2(1) \\
&= \beta_1(\beta_1 + 1)f_2(1), \\
f_2(1) &= \frac{1}{\beta_1(\beta_1 + 1)} \frac{\Gamma(2 - \beta_1)\Gamma(1 - \beta_1)}{\Gamma\left(\frac{1}{2} - \beta_1 - \nu\right)\Gamma\left(\frac{1}{2} - \beta_1 + \nu\right)} \\
&= \frac{\Gamma(2 - \beta_1)\Gamma(-1 - \beta_1)}{\Gamma\left(\frac{1}{2} - \beta_1 - \nu\right)\Gamma\left(\frac{1}{2} - \beta_1 + \nu\right)}.
\end{aligned}$$

Therefore, we can expand $F_1(z)$ near $z = 1$ as

$$\begin{aligned}
F_1(z) &= f_1(z) + (1 - z)^{1+\beta_1} f_2(z) \\
&= \frac{\Gamma(2 - \beta_1)\Gamma(1 + \beta_1)}{\Gamma\left(\frac{3}{2} - \nu\right)\Gamma\left(\frac{3}{2} + \nu\right)} \\
&\quad - \left(\frac{1}{2} - \beta_1 - \nu\right) \left(\frac{1}{2} - \beta_1 + \nu\right) \frac{\Gamma(2 - \beta_1)\Gamma(\beta_1)}{\Gamma\left(\frac{3}{2} - \nu\right)\Gamma\left(\frac{3}{2} + \nu\right)} (1 - z) \\
&\quad + \frac{\Gamma(2 - \beta_1)\Gamma(-1 - \beta_1)}{\Gamma\left(\frac{1}{2} - \beta_1 - \nu\right)\Gamma\left(\frac{1}{2} - \beta_1 + \nu\right)} (1 - z)^{1+\beta_1} + O(1 - z)^2.
\end{aligned}$$

Naturally,

$$\begin{aligned}
F'_1(z) &= \left(\frac{1}{2} - \beta_1 - \nu\right) \left(\frac{1}{2} - \beta_1 + \nu\right) \frac{\Gamma(2 - \beta_1)\Gamma(\beta_1)}{\Gamma\left(\frac{3}{2} - \nu\right)\Gamma\left(\frac{3}{2} + \nu\right)} \\
&\quad + \frac{\Gamma(2 - \beta_1)\Gamma(-\beta_1)}{\Gamma\left(\frac{1}{2} - \beta_1 - \nu\right)\Gamma\left(\frac{1}{2} - \beta_1 + \nu\right)} (1 - z)^{\beta_1} + O(1 - z).
\end{aligned}$$

As for $F_2(z)$, by Euler transform we have

$$F_2(z) = (1-z)^{-1-\beta_1} F\left(-\frac{1}{2}-\nu, -\frac{1}{2}+\nu; \beta_1; z\right).$$

Similarly, we can expand $F(-\frac{1}{2}-\nu, -\frac{1}{2}+\nu; \beta_1; z)$ near $z = 1$:

$$\begin{aligned} & F\left(-\frac{1}{2}-\nu, -\frac{1}{2}+\nu; \beta_1; z\right) \\ = & \frac{\Gamma(\beta_1)\Gamma(\beta_1+1)}{\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} \\ & - \left(-\frac{1}{2}-\nu\right)\left(-\frac{1}{2}+\nu\right) \frac{\Gamma(\beta_1)\Gamma(\beta_1)}{\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} (1-z) \\ & + \frac{\Gamma(\beta_1)\Gamma(-1-\beta_1)}{\Gamma(-\frac{1}{2}-\nu)\Gamma(-\frac{1}{2}+\nu)} (1-z)^{1+\beta_1} + O(1-z)^2, \end{aligned}$$

hence

$$\begin{aligned} F_2(z) &= (1-z)^{-1-\beta_1} F\left(-\frac{1}{2}-\nu, -\frac{1}{2}+\nu; \beta_1; z\right) \\ &= \frac{\Gamma(\beta_1)\Gamma(\beta_1+1)}{\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} (1-z)^{-1-\beta_1} \\ &\quad - \left(-\frac{1}{2}-\nu\right)\left(-\frac{1}{2}+\nu\right) \frac{\Gamma(\beta_1)\Gamma(\beta_1)}{\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} (1-z)^{-\beta_1} \\ &\quad + \frac{\Gamma(\beta_1)\Gamma(-1-\beta_1)}{\Gamma(-\frac{1}{2}-\nu)\Gamma(-\frac{1}{2}+\nu)} (1-z)^{1-\beta_1} + O(1-z)^{1-\beta_1}, \\ F'_2(z) &= \frac{\Gamma(\beta_1)\Gamma(\beta_1+2)}{\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} (1-z)^{-2-\beta_1} \\ &\quad - \left(-\frac{1}{2}-\nu\right)\left(-\frac{1}{2}+\nu\right) \frac{\Gamma(\beta_1)\Gamma(\beta_1+1)}{\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} (1-z)^{-1-\beta_1} + O(1-z)^{-\beta_1}. \end{aligned}$$

Therefore,

$$\begin{aligned}
F_1(z)F_2(z) &= \left[\frac{\Gamma(2-\beta_1)\Gamma(1+\beta_1)}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)} \right. \\
&\quad - \left(\frac{1}{2} - \beta_1 - \nu \right) \left(\frac{1}{2} - \beta_1 + \nu \right) \frac{\Gamma(2-\beta_1)\Gamma(\beta_1)}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)} (1-z) \\
&\quad + \frac{\Gamma(2-\beta_1)\Gamma(-1-\beta_1)}{\Gamma(\frac{1}{2}-\beta_1-\nu)\Gamma(\frac{1}{2}-\beta_1+\nu)} (1-z)^{1+\beta_1} + O(1-z)^2 \Big] \\
&\quad \times \left[\frac{\Gamma(\beta_1)\Gamma(\beta_1+1)}{\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} (1-z)^{-1-\beta_1} \right. \\
&\quad - \left(-\frac{1}{2} - \nu \right) \left(-\frac{1}{2} + \nu \right) \frac{\Gamma(\beta_1)\Gamma(\beta_1)}{\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} (1-z)^{-\beta_1} \\
&\quad + \frac{\Gamma(\beta_1)\Gamma(-1-\beta_1)}{\Gamma(-\frac{1}{2}-\nu)\Gamma(-\frac{1}{2}+\nu)} + O(1-z)^{1-\beta_1} \Big] \\
&= \frac{\Gamma(2-\beta_1)\Gamma(1+\beta_1)\Gamma(\beta_1)\Gamma(\beta_1+1)}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} (1-z)^{-1-\beta_1} \\
&\quad - \frac{(\frac{1}{2}-\nu)(\frac{1}{2}+\nu)\Gamma(2-\beta_1)\Gamma(1+\beta_1)\Gamma(\beta_1)\Gamma(\beta_1)}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} (1-z)^{-\beta_1} \\
&\quad + \frac{\Gamma(2-\beta_1)\Gamma(1+\beta_1)\Gamma(\beta_1)\Gamma(-1-\beta_1)}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)\Gamma(-\frac{1}{2}-\nu)\Gamma(-\frac{1}{2}+\nu)} \\
&\quad - \frac{(\frac{1}{2}-\beta_1-\nu)(\frac{1}{2}-\beta_1+\nu)\Gamma(2-\beta_1)\Gamma(\beta_1)\Gamma(\beta_1)\Gamma(\beta_1+1)}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} (1-z)^{-\beta_1} \\
&\quad + \frac{\Gamma(2-\beta_1)\Gamma(-1-\beta_1)\Gamma(\beta_1)\Gamma(\beta_1+1)}{\Gamma(\frac{1}{2}-\beta_1-\nu)\Gamma(\frac{1}{2}-\beta_1+\nu)\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} \\
&\quad + O(1-z)^{1-\beta_1},
\end{aligned}$$

$$\begin{aligned}
F_1(z)F_2'(z) &= \left[\frac{\Gamma(2-\beta_1)\Gamma(1+\beta_1)}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)} \right. \\
&\quad - \left(\frac{1}{2} - \beta_1 - \nu \right) \left(\frac{1}{2} - \beta_1 + \nu \right) \frac{\Gamma(2-\beta_1)\Gamma(\beta_1)}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)} (1-z) \\
&\quad \left. + \frac{\Gamma(2-\beta_1)\Gamma(-1-\beta_1)}{\Gamma(\frac{1}{2}-\beta_1-\nu)\Gamma(\frac{1}{2}-\beta_1+\nu)} (1-z)^{1+\beta_1} + O(1-z)^2 \right] \\
&\quad \times \left[\frac{\Gamma(\beta_1)\Gamma(\beta_1+2)}{\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} (1-z)^{-2-\beta_1} \right. \\
&\quad \left. - \frac{(-\frac{1}{2}-\nu)(-\frac{1}{2}+\nu)\Gamma(\beta_1)\Gamma(\beta_1+1)}{\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} (1-z)^{-1-\beta_1} + O(1-z)^{-\beta_1} \right] \\
&= \frac{\Gamma(2-\beta_1)\Gamma(1+\beta_1)\Gamma(\beta_1)\Gamma(\beta_1+2)}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} (1-z)^{-2-\beta_1} \\
&\quad - \frac{(-\frac{1}{2}-\nu)(-\frac{1}{2}+\nu)\Gamma(2-\beta_1)\Gamma(1+\beta_1)\Gamma(\beta_1)\Gamma(\beta_1+1)}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} (1-z)^{-1-\beta_1} \\
&\quad - \frac{(\frac{1}{2}-\beta_1-\nu)(\frac{1}{2}-\beta_1+\nu)\Gamma(2-\beta_1)\Gamma(\beta_1)\Gamma(\beta_1)\Gamma(\beta_1+2)}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} (1-z)^{-1-\beta_1} \\
&\quad + \frac{\Gamma(2-\beta_1)\Gamma(-1-\beta_1)\Gamma(\beta_1)\Gamma(\beta_1+2)}{\Gamma(\frac{1}{2}-\beta_1-\nu)\Gamma(\frac{1}{2}-\beta_1+\nu)\Gamma(\beta_1+\frac{1}{2}-\nu)\Gamma(\beta_1+\frac{1}{2}+\nu)} (1-z)^{-1} \\
&\quad + O(1-z)^{-\beta_1},
\end{aligned}$$

$$\begin{aligned}
F_1'(z)F_2(z) &= \left[\left(\frac{1}{2} - \beta_1 - \nu \right) \left(\frac{1}{2} - \beta_1 + \nu \right) \frac{\Gamma(2 - \beta_1)\Gamma(\beta_1)}{\Gamma\left(\frac{3}{2} - \nu\right)\Gamma\left(\frac{3}{2} + \nu\right)} \right. \\
&\quad \left. + \frac{\Gamma(2 - \beta_1)\Gamma(-\beta_1)}{\Gamma\left(\frac{1}{2} - \beta_1 - \nu\right)\Gamma\left(\frac{1}{2} - \beta_1 + \nu\right)} (1 - z)^{\beta_1} + O(1 - z) \right] \\
&\quad \times \left[\frac{\Gamma(\beta_1)\Gamma(\beta_1 + 1)}{\Gamma\left(\beta_1 + \frac{1}{2} - \nu\right)\Gamma\left(\beta_1 + \frac{1}{2} + \nu\right)} (1 - z)^{-1 - \beta_1} \right. \\
&\quad \left. - \left(-\frac{1}{2} - \nu \right) \left(-\frac{1}{2} + \nu \right) \frac{\Gamma(\beta_1)\Gamma(\beta_1)}{\Gamma\left(\beta_1 + \frac{1}{2} - \nu\right)\Gamma\left(\beta_1 + \frac{1}{2} + \nu\right)} (1 - z)^{-\beta_1} \right. \\
&\quad \left. + \frac{\Gamma(\beta_1)\Gamma(-1 - \beta_1)}{\Gamma\left(-\frac{1}{2} - \nu\right)\Gamma\left(-\frac{1}{2} + \nu\right)} + O(1 - z)^{1 - \beta_1} \right] \\
&= \frac{\left(\frac{1}{2} - \beta_1 - \nu \right) \left(\frac{1}{2} - \beta_1 + \nu \right) \Gamma(2 - \beta_1)\Gamma(\beta_1)\Gamma(\beta_1)\Gamma(\beta_1 + 1)}{\Gamma\left(\frac{3}{2} - \nu\right)\Gamma\left(\frac{3}{2} + \nu\right)\Gamma\left(\beta_1 + \frac{1}{2} - \nu\right)\Gamma\left(\beta_1 + \frac{1}{2} + \nu\right)} (1 - z)^{-1 - \beta_1} \\
&\quad + \frac{\Gamma(2 - \beta_1)\Gamma(-\beta_1)\Gamma(\beta_1)\Gamma(\beta_1 + 1)}{\Gamma\left(\frac{1}{2} - \beta_1 - \nu\right)\Gamma\left(\frac{1}{2} - \beta_1 + \nu\right)\Gamma\left(\beta_1 + \frac{1}{2} - \nu\right)\Gamma\left(\beta_1 + \frac{1}{2} + \nu\right)} (1 - z)^{-1} \\
&\quad + O(1 - z)^{-\beta_1}.
\end{aligned}$$

By $\Gamma(-1 - \beta_1)\Gamma(\beta_1 + 2) = (1 + \beta_1)\Gamma(-1 - \beta_1)\Gamma(\beta_1 + 1) = -\Gamma(-\beta_1)\Gamma(\beta_1 + 1)$,
we have

$$\begin{aligned}
&F_1(z)F_2'(z) - F_1'(z)F_2(z) \\
&= \frac{\Gamma(2 - \beta_1)\Gamma(1 + \beta_1)\Gamma(\beta_1)\Gamma(\beta_1 + 2)}{\Gamma\left(\frac{3}{2} - \nu\right)\Gamma\left(\frac{3}{2} + \nu\right)\Gamma\left(\beta_1 + \frac{1}{2} - \nu\right)\Gamma\left(\beta_1 + \frac{1}{2} + \nu\right)} (1 - z)^{-2 - \beta_1} \\
&\quad - \frac{\left(-\frac{1}{2} - \nu \right) \left(-\frac{1}{2} + \nu \right) \Gamma(2 - \beta_1)\Gamma(1 + \beta_1)\Gamma(\beta_1)\Gamma(\beta_1 + 1)}{\Gamma\left(\frac{3}{2} - \nu\right)\Gamma\left(\frac{3}{2} + \nu\right)\Gamma\left(\beta_1 + \frac{1}{2} - \nu\right)\Gamma\left(\beta_1 + \frac{1}{2} + \nu\right)} (1 - z)^{-1 - \beta_1} \\
&\quad - \frac{\left(\frac{1}{2} - \beta_1 - \nu \right) \left(\frac{1}{2} - \beta_1 + \nu \right) \Gamma(2 - \beta_1)\Gamma(\beta_1)\Gamma(\beta_1)(\beta_1 + 2)\Gamma(\beta_1 + 1)}{\Gamma\left(\frac{3}{2} - \nu\right)\Gamma\left(\frac{3}{2} + \nu\right)\Gamma\left(\beta_1 + \frac{1}{2} - \nu\right)\Gamma\left(\beta_1 + \frac{1}{2} + \nu\right)} (1 - z)^{-1 - \beta_1} \\
&\quad + \frac{2\Gamma(2 - \beta_1)\Gamma(-1 - \beta_1)\Gamma(\beta_1)\Gamma(\beta_1 + 2)}{\Gamma\left(\frac{1}{2} - \beta_1 - \nu\right)\Gamma\left(\frac{1}{2} - \beta_1 + \nu\right)\Gamma\left(\beta_1 + \frac{1}{2} - \nu\right)\Gamma\left(\beta_1 + \frac{1}{2} + \nu\right)} (1 - z)^{-1} \\
&\quad + O(1 - z)^{-\beta_1},
\end{aligned}$$

Hence

$$\begin{aligned}
H(z) &= (1 - \beta_1 + 2z\beta_1) F_1(z)F_2(z) - z(1 - z) (F_1(z)F_2'(z) - F_1'(z)F_2(z)) \\
&= (1 + \beta_1 - 2(1 - z)\beta_1) F_1(z)F_2(z) + [(1 - z) - 1] (1 - z) (F_1(z)F_2'(z) - F_1'(z)F_2(z)) \\
&= C_1(1 - z)^{-1 - \beta_1} + C_2(1 - z)^{-\beta_1} + C_3 + O(1 - z)^{1 - \beta_1}
\end{aligned}$$

$$\begin{aligned}
C_1 &= (1 + \beta_1) \frac{\Gamma(2 - \beta_1) \Gamma(1 + \beta_1) \Gamma(\beta_1) \Gamma(\beta_1 + 1)}{\Gamma(\frac{3}{2} - \nu) \Gamma(\frac{3}{2} + \nu) \Gamma(\beta_1 + \frac{1}{2} - \nu) \Gamma(\beta_1 + \frac{1}{2} + \nu)} \\
&\quad - \frac{\Gamma(2 - \beta_1) \Gamma(1 + \beta_1) \Gamma(\beta_1) \Gamma(\beta_1 + 2)}{\Gamma(\frac{3}{2} - \nu) \Gamma(\frac{3}{2} + \nu) \Gamma(\beta_1 + \frac{1}{2} - \nu) \Gamma(\beta_1 + \frac{1}{2} + \nu)} \\
&= 0, \\
C_2 &= -2\beta_1 \frac{\Gamma(2 - \beta_1) \Gamma(1 + \beta_1) \Gamma(\beta_1) \Gamma(\beta_1 + 1)}{\Gamma(\frac{3}{2} - \nu) \Gamma(\frac{3}{2} + \nu) \Gamma(\beta_1 + \frac{1}{2} - \nu) \Gamma(\beta_1 + \frac{1}{2} + \nu)} \\
&\quad - (1 + \beta_1) \frac{(\frac{1}{2} - \nu) (\frac{1}{2} + \nu) \Gamma(2 - \beta_1) \Gamma(1 + \beta_1) \Gamma(\beta_1) \Gamma(\beta_1)}{\Gamma(\frac{3}{2} - \nu) \Gamma(\frac{3}{2} + \nu) \Gamma(\beta_1 + \frac{1}{2} - \nu) \Gamma(\beta_1 + \frac{1}{2} + \nu)} \\
&\quad - (1 + \beta_1) \frac{(\frac{1}{2} - \beta_1 - \nu) (\frac{1}{2} - \beta_1 + \nu) \Gamma(2 - \beta_1) \Gamma(\beta_1) \Gamma(\beta_1) \Gamma(\beta_1 + 1)}{\Gamma(\frac{3}{2} - \nu) \Gamma(\frac{3}{2} + \nu) \Gamma(\beta_1 + \frac{1}{2} - \nu) \Gamma(\beta_1 + \frac{1}{2} + \nu)} \\
&\quad + \frac{\Gamma(2 - \beta_1) \Gamma(1 + \beta_1) \Gamma(\beta_1) \Gamma(\beta_1 + 2)}{\Gamma(\frac{3}{2} - \nu) \Gamma(\frac{3}{2} + \nu) \Gamma(\beta_1 + \frac{1}{2} - \nu) \Gamma(\beta_1 + \frac{1}{2} + \nu)} \\
&\quad + \frac{(-\frac{1}{2} - \nu) (-\frac{1}{2} + \nu) \Gamma(2 - \beta_1) \Gamma(1 + \beta_1) \Gamma(\beta_1) \Gamma(\beta_1 + 1)}{\Gamma(\frac{3}{2} - \nu) \Gamma(\frac{3}{2} + \nu) \Gamma(\beta_1 + \frac{1}{2} - \nu) \Gamma(\beta_1 + \frac{1}{2} + \nu)} \\
&\quad + \frac{(\frac{1}{2} - \beta_1 - \nu) (\frac{1}{2} - \beta_1 + \nu) \Gamma(2 - \beta_1) \Gamma(\beta_1) \Gamma(\beta_1) (\beta_1 + 2) \Gamma(\beta_1 + 1)}{\Gamma(\frac{3}{2} - \nu) \Gamma(\frac{3}{2} + \nu) \Gamma(\beta_1 + \frac{1}{2} - \nu) \Gamma(\beta_1 + \frac{1}{2} + \nu)} \\
&= \frac{(-\beta_1 + 1) \Gamma(2 - \beta_1) \Gamma(1 + \beta_1) \Gamma(\beta_1) \Gamma(\beta_1 + 1)}{\Gamma(\frac{3}{2} - \nu) \Gamma(\frac{3}{2} + \nu) \Gamma(\beta_1 + \frac{1}{2} - \nu) \Gamma(\beta_1 + \frac{1}{2} + \nu)} \\
&\quad - \frac{(\frac{1}{2} - \nu) (\frac{1}{2} + \nu) \Gamma(2 - \beta_1) \Gamma(1 + \beta_1) \Gamma(\beta_1) \Gamma(\beta_1)}{\Gamma(\frac{3}{2} - \nu) \Gamma(\frac{3}{2} + \nu) \Gamma(\beta_1 + \frac{1}{2} - \nu) \Gamma(\beta_1 + \frac{1}{2} + \nu)} \\
&\quad + \frac{(\frac{1}{2} - \beta_1 - \nu) (\frac{1}{2} - \beta_1 + \nu) \Gamma(2 - \beta_1) \Gamma(\beta_1) \Gamma(\beta_1) \Gamma(\beta_1 + 1)}{\Gamma(\frac{3}{2} - \nu) \Gamma(\frac{3}{2} + \nu) \Gamma(\beta_1 + \frac{1}{2} - \nu) \Gamma(\beta_1 + \frac{1}{2} + \nu)} \\
&= \left[(-\beta_1 + 1) \beta_1 - \left(\frac{1}{2} - \nu \right) \left(\frac{1}{2} + \nu \right) + \left(\frac{1}{2} - \beta_1 - \nu \right) \left(\frac{1}{2} - \beta_1 + \nu \right) \right] \\
&\quad \times \frac{\Gamma(2 - \beta_1) \Gamma(\beta_1) \Gamma(\beta_1) \Gamma(\beta_1 + 1)}{\Gamma(\frac{3}{2} - \nu) \Gamma(\frac{3}{2} + \nu) \Gamma(\beta_1 + \frac{1}{2} - \nu) \Gamma(\beta_1 + \frac{1}{2} + \nu)} \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
C_3 &= (1 + \beta_1) \frac{\Gamma(2 - \beta_1)\Gamma(1 + \beta_1)\Gamma(\beta_1)\Gamma(-1 - \beta_1)}{\Gamma(\frac{3}{2} - \nu)\Gamma(\frac{3}{2} + \nu)\Gamma(-\frac{1}{2} - \nu)\Gamma(-\frac{1}{2} + \nu)} \\
&\quad + (1 + \beta_1) \frac{\Gamma(2 - \beta_1)\Gamma(-1 - \beta_1)\Gamma(\beta_1)\Gamma(\beta_1 + 1)}{\Gamma(\frac{1}{2} - \beta_1 - \nu)\Gamma(\frac{1}{2} - \beta_1 + \nu)\Gamma(\beta_1 + \frac{1}{2} - \nu)\Gamma(\beta_1 + \frac{1}{2} + \nu)} \\
&\quad - \frac{2\Gamma(2 - \beta_1)\Gamma(-1 - \beta_1)\Gamma(\beta_1)\Gamma(\beta_1 + 2)}{\Gamma(\frac{1}{2} - \beta_1 - \nu)\Gamma(\frac{1}{2} - \beta_1 + \nu)\Gamma(\beta_1 + \frac{1}{2} - \nu)\Gamma(\beta_1 + \frac{1}{2} + \nu)} \\
&= \frac{\Gamma(2 - \beta_1)\Gamma(2 + \beta_1)\Gamma(\beta_1)\Gamma(-1 - \beta_1)}{\Gamma(\frac{3}{2} - \nu)\Gamma(\frac{3}{2} + \nu)\Gamma(-\frac{1}{2} - \nu)\Gamma(-\frac{1}{2} + \nu)} \\
&\quad - \frac{\Gamma(2 - \beta_1)\Gamma(-1 - \beta_1)\Gamma(\beta_1)\Gamma(\beta_1 + 2)}{\Gamma(\frac{1}{2} - \beta_1 - \nu)\Gamma(\frac{1}{2} - \beta_1 + \nu)\Gamma(\beta_1 + \frac{1}{2} - \nu)\Gamma(\beta_1 + \frac{1}{2} + \nu)} \\
&= (\beta_1 - 1) \frac{\Gamma(2 - \beta_1)\Gamma(-1 + \beta_1)\Gamma(2 + \beta_1)\Gamma(-1 - \beta_1)}{\Gamma(\frac{3}{2} - \nu)\Gamma(-\frac{1}{2} + \nu)\Gamma(\frac{3}{2} + \nu)\Gamma(-\frac{1}{2} - \nu)} \\
&\quad - (\beta_1 - 1) \frac{\Gamma(2 - \beta_1)\Gamma(-1 + \beta_1)\Gamma(2 + \beta_1)\Gamma(-1 - \beta_1)}{\Gamma(\frac{1}{2} - \beta_1 - \nu)\Gamma(\beta_1 + \frac{1}{2} + \nu)\Gamma(\frac{1}{2} - \beta_1 + \nu)\Gamma(\beta_1 + \frac{1}{2} - \nu)} \\
&= \frac{(\beta_1 - 1) \left(\sin\left(\left(\frac{3}{2} - \nu\right)\pi\right) \sin\left(\left(\frac{3}{2} + \nu\right)\pi\right) - \sin\left(\left(\frac{1}{2} - \beta_1 - \nu\right)\pi\right) \sin\left(\left(\frac{1}{2} - \beta_1 + \nu\right)\pi\right) \right)}{\sin((2 - \beta_1)\pi) \sin((2 + \beta_1)\pi)} \\
&= \frac{(\beta_1 - 1) (\cos(\nu\pi) \cos(\beta_1\pi) - \cos((\beta_1 - \nu)\pi) \cos((\beta_1 + \nu)\pi))}{-\sin(\beta_1\pi) \sin(\beta_1\pi)} \\
&= (\beta_1 - 1) \frac{\cos^2(\nu\pi) - \cos^2(\beta_1\pi) \cos^2(\nu\pi) + \sin^2(\beta_1\pi) \sin^2(\nu\pi)}{-\sin(\beta_1\pi) \sin(\beta_1\pi)} \\
&= 1 - \beta_1.
\end{aligned}$$

Therefore, $H(z) = 1 - \beta_1 + O(1 - z)^{1 - \beta_1}$, so $\lim_{z \rightarrow 1} H(z) = 1 - \beta_1$.

As for $\lim_{z \rightarrow \infty} H(z)$, we can use the Pfaff transform (17-18) to expand $F_1(z), F_2(z)$ at infinity, since we can determine the power series of its transformed function at $\frac{z}{z-1} = 1$. By a similar calculation, we can eventually prove

$$H(z) = 1 - \beta_1 + O\left(\frac{1}{z}\right)$$

in a neighborhood of $z = \infty$. Hence $\lim_{z \rightarrow \infty} H(z) = 1 - \beta_1$.

Since $H(z)$ is holomorphic elsewhere, $H(z)$ must be a constant by maximum modulus theorem, provided $H(z)$ is a single-valued analytic function. However, this is not the case. $z = 1$ is a possible branch point, and maximum modulus theorem cannot be applied on multivalued functions. In order to overcome this problem, we construct a single-valued function based on $H(z)$.

By series expansion in a neighborhood of $z = 1$, $F_1(z)$ and $F_2(z)$ can be expressed as

$$\begin{aligned}
F_1(z) &= h_1(z) + (1 - z)^{1 + \beta_1} h_2(z), \\
F_2(z) &= h_3(z) + (1 - z)^{-1 - \beta_1} h_4(z).
\end{aligned}$$

where h_i is some holomorphic function in the neighborhood. Therefore, $H(z)$ near $z = 1$ can be expressed as

$$\begin{aligned} H(z) &= (1 - \beta_1 + 2z\beta_1) F_1(z)F_2(z) - z(1 - z) (F_1(z)F_2'(z) - F_1'(z)F_2(z)) \\ &= h_5(z) + (1 - z)^{1+\beta_1} h_6(z) + (1 - z)^{-1-\beta_1} h_7(z) \end{aligned}$$

for some analytic h_5, h_6, h_7 .

We first consider that β_1 is a rational number $\frac{q}{p}$. In this case, $z = 1$ is an algebraic branch point. Define $s(z) = 1 - (1 - z)^p$, then

$$H(s(z)) = h_5(1 - (1 - z)^p) + (1 - z)^{p+q} h_6(1 - (1 - z)^p) + (1 - z)^{-p-q} h_7(1 - (1 - z)^p)$$

is a meromorphic function in a neighborhood of $z = 1$. But since $H(s(1)) = H(1) = 1 - \beta_1$ is a finite number, we know $z = 1$ is not a poll, so $H \circ s$ must be analytic in the whole plane. Notice that

$$\lim_{z \rightarrow \infty} H(s(z)) = \lim_{s \rightarrow \infty} H(s) = 1 - \beta_1$$

is bounded at infinity. By Liouville theorem, $H \circ s$ must be a constant function, so $H(s(z)) = 1 - \beta_1$. Hence $H(z) = 1 - \beta_1$ for all $z \in \mathbb{C}$.

If β_1 is irrational, $z = 1$ is not an algebraic branch point, but a logarithmic branch point. The former argument is no longer valid. However, since rational numbers are dense in $(0, 1)$, $H(\nu, \beta_1; z)$ for $|z| < 1$ must also be $1 - \beta_1$, because by definition $H(\nu, \beta_1; z)$ continuously depends on β_1 in this region. By analyticity, $H(z)$ must also be $1 - \beta_1$ in $\mathbb{C} \setminus \{1\}$. $H(1) = 1 - \beta_1$ follows by continuity. ■

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